COMPUTING BASIS OF SOLUTIONS OF ANY MAHLER EQUATION

Abstract. Mahler equations arise in a wide range of contexts including the study of finite automata, regular sequences, algebraic series over $\mathbb{F}_p(z)$, and periods of Drinfeld modules. Introduced a century ago by K. Mahler to study the transcendence of certain complex numbers, they have recently been the subject of several works establishing a deep connection between such transcendence properties and the nature of their solutions. While numerous studies have investigated these solutions, existing algorithms can only compute them in specific rings: rational functions, power series, Puiseux series, or Hahn series. This paper solves the problem by providing an algorithm that computes a complete basis of solutions for any Mahler equation, along with a decomposition of each solution over the field of Puiseux series. Along the way, we describe an algorithm that computes a fundamental matrix of solutions for any Mahler system.

Contents

1. Introduction

The study of solutions of linear differential equations is a classical and extensive field of research. Any linear differential equation of order m with coefficients in $\mathbb{C}((z))$ admits a basis of solutions consisting of $\mathbb{C}$-linear combinations of elements of the form

$$f(z)z^{\alpha}(\log z)^k e^{P(z^{-1/m!})} \tag{1}$$

where $f(z)$ is a Gevrey series, $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, and $P \in X\mathbb{C}[X]$. Various algorithms have been established for computing such solutions [Tur55, Tou87, Bar97].

The subject of our paper is the so-called Mahler equations. A linear Mahler equation is an equation of the form

$$a_0(z)y(z) + a_1(z)y(z^p) + \cdots + a_m(z)y(z^{p^m}) = 0$$

where $p \geq 2$ is an integer, $a_0 a_m \neq 0$ and $a_0, \ldots, a_m$ are, depending on the context, polynomials, power series, etc. Though less widely known than their differential counterparts, they have been the subject of numerous recent papers due to their connections with several areas, see the discussion in Section 2. Only quite recently, a general form for a basis of solutions analogous to (1) has been established for Mahler equations

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(see [FR25]). However, the proof is nonconstructive. Algorithms do exist to compute rational, power series, Puiseux series [CDDM18], or Hahn series [FR24] solutions of a given Mahler equation, but none can provide a full basis of solutions. Our paper fills this gap.

1.1. **Linear Mahler equations.** Let $\mathbf{K}$ be a field and z be an indeterminate. For each integer $p \geq 2$, we consider the map $\phi_p : \mathbf{K}(z) \to \mathbf{K}(z)$ defined by $z \mapsto z^p$. Let $\mathcal{R}$ be a ring extension of $\mathbf{K}(z)$ and let ϕ_p extend to $\mathcal{R}$. A p-Mahler equation over $\mathcal{R}$ is a linear functional equation of the form

$$a_0 y + a_1 \phi_p(y) + \cdots + a_m \phi_p^m(y) = 0 \tag{2}$$

where $a_0, \ldots, a_m \in \mathcal{R}$ and $a_0 a_m \neq 0$. In what follows, we focus on the case where $\mathcal{R} = \mathbf{K}[[z]]$. However, the methods we develop here extend immediately to the case where $\mathcal{R}$ is the field of Puiseux series over $\mathbf{K}$.

Power series –or even Puiseux series– are not sufficient to solve arbitrary Mahler equations over $\mathbf{K}[[z]]$. A famous example comes from the equation

$$y(z^p) - y(z) = z^{-1}$$

for which a quick computation reveals that it has no Puiseux series solution. Meanwhile, the formal series $h(z) = \sum_{k \geq 1} z^{-1/p^k}$ is a solution to this equation. This is not a Puiseux series, but since its support is a well-ordered subset of the rational numbers, $h(z)$ is a *Hahn series*. Generalizing this remark, in [FR25], the authors proved that[1], given an integer $s \geq 1$, a s-dimensional multi-linear recurrence sequence $\boldsymbol{u} = (u_{k_1,\ldots,k_s})_{k_1,\ldots,k_s} \in \overline{\mathbf{K}}^{\mathbb{Z}^s}$ and a s-tuple of positive rational numbers $\boldsymbol{a} = (a_1, \ldots, a_s)$, the Hahn series

$$\boldsymbol{\xi}_{\boldsymbol{\omega}} := \sum_{k_1,\cdots,k_s \geq 1} u_{k_1,\ldots,k_s} z^{-\frac{a_1}{p^{k_1}} - \cdots - \frac{a_s}{p^{k_1+\cdots+k_s}}} \quad \text{with } \boldsymbol{\omega} := (\boldsymbol{u}, \boldsymbol{a}) \tag{3}$$

is a solution to some Mahler equation over $\overline{\mathbf{K}}[[z]]$, where we let $\overline{\mathbf{K}}$ denote an algebraic closure of $\mathbf{K}$. We let Ω denote the set of all such tuples $\boldsymbol{\omega}$. For convenience, when $s = 0$ we let $\boldsymbol{\xi}_{\boldsymbol{\omega}} = 1$. The definition and some details about linear recurrent sequences are given in Section 5.1. When $\overline{\mathbf{K}}$ has characteristic 0, any linear recurrent sequence can be uniquely written as a linear combination of sequences with general term

$$k_1^{\alpha_1} \cdots k_s^{\alpha_s} \lambda_1^{k_1} \cdots \lambda_s^{k_s}, \quad \alpha_1, \ldots, \alpha_s \in \mathbb{Z}_{\geq 0},\ \lambda_1, \ldots, \lambda_s \in \overline{\mathbf{K}}^{\times}. \tag{4}$$

Hence, we recover the Hahn series introduced in [FR25]. Such a decomposition no longer holds when $\overline{\mathbf{K}}$ has positive characteristic.

Unfortunately, Hahn series fail to provide full basis of solutions of any Mahler equations. The equations $\phi_p(y) = cy$ with $c \in \overline{\mathbf{K}}^{\times} \setminus \{1\}$ or $\phi_p(y) = y + 1$ have no solutions in the field of Hahn series, as one can check by looking at the hypothetical valuation of such a solution. However, it is possible to find solutions to these equations in some formal ring extension of the field of Hahn series. We denote by e_c a solution to the former and by ℓ a solution to the latter, in such an extension. For convenience, we also let $e_1 = 1$. As mentioned in [FP25, Section 2.2], one can construct such a ring

[1] While the results in [FR25] are stated for $\overline{\mathbf{K}} := \overline{\mathbb{Q}}$, those invoked here hold over the algebraic closure of any field. Since we must reprove these results for algorithmic considerations, we provide details later in the paper.

extension with field of constants $\{f : \varphi_p(f) = f\} = \overline{\mathbf{K}}$. Then, it follows from [FR25] that any Mahler equation over $\mathbf{K}[[z]]$ has a basis of solutions[2] $y_1, \dots, y_m$ of the form

$$y_i = \sum_{c \in \overline{\mathbf{K}}^\times} \sum_{j \in \mathbb{Z}_{\geq 0}} \sum_{\boldsymbol{\omega} \in \Omega} f_{i,c,j,\boldsymbol{\omega}}(z) \boldsymbol{\xi}_{\boldsymbol{\omega}} e_c \ell^j \tag{5}$$

where the $f_{i,c,j,\boldsymbol{\omega}}$ are Puiseux series with coefficients in $\overline{\mathbf{K}}$ and each sum has finite support. When $\mathbf{K}$ is of characteristic 0, this decomposition is unique if one restricts $\boldsymbol{\omega}$ to the set $\Omega_0 \subset \Omega$ of tuples $\boldsymbol{\omega} = (\boldsymbol{u}, \boldsymbol{a})$ for which $\boldsymbol{u}$ is of the form (4) and the entries of $\boldsymbol{a}$ have denominators coprime with p and numerators not divisible by p, see [FR25]. The aim of this article is to provide an algorithm to compute such a basis of solutions.

Theorem 1. *Let $\mathbf{K}$ be an effective field of characteristic 0. Algorithm 4 takes as input a Mahler equation[3] over $\mathbf{K}[[z]]$ of the form (2) and returns a basis $y_1, \dots, y_m$ of solutions of (2) under the form (5). Precisely, it returns:*

- *a finite set $K_0 \subset \overline{\mathbf{K}}^\times$, an integer $j_0 \geq 0$, and a finite set $\Omega_1 \subset \Omega_0$ such that the supports of the sums (5) are included in $K_0 \times \{0, \dots, j_0\} \times \Omega_1$,*
- *an integer d and an integer v such that $f_{i,c,j,\boldsymbol{\omega}}(z) \in z^{-v}\overline{\mathbf{K}}[[z^{1/d}]]$ for every $i, c, j, \boldsymbol{\omega}$,*
- *the coefficients of the Puiseux expansion of each $f_{i,c,j,\boldsymbol{\omega}}(z)$ up to any prescribed order.*

Remark. (i) The nonzero Puiseux series $f_{i,c,j,\boldsymbol{\omega}}(z)$ appearing in (5) are also solutions of some p-Mahler equations. Such Puiseux series are uniquely determined by their associated equation and their first coefficients. It is possible to adapt Algorithm 4 to associate to each $f_{i,c,j,\boldsymbol{\omega}}(z)$ a p-Mahler equation it satisfies. In particular, this allows one to check whether such a series is 0 or whether there are linear relations, or algebraic relations of a given degree, between them (see [AF17]).

(ii) Our algorithm can be used to compute solutions in specific rings, such as power, Puiseux or Hahn series. While it does not improve the existing algorithm for Puiseux series, it significantly improves the one for Hahn series. Indeed, the algorithm in [FR24] computes the coefficients of z^γ of any Hahn series solution for exponents γ inside a given finite set. Using Theorem 1, given an integer N, we can provide a closed formula for the coefficients of z^γ with $|\gamma| \leq N$, which is an infinite set.

1.2. **Linear Mahler systems.** Recall that we let ϕ_p denote the map $z \mapsto z^p$. Although Theorem 1 is about linear equations, the algorithm it refers to relies on our ability to solve linear systems. A p-Mahler system over $\mathbf{K}((z))$ is a system of the form

$$\phi_p(Y) = AY \tag{6}$$

with $A \in \mathrm{GL}_m(\mathbf{K}((z)))$. Let $\mathscr{P}_{\mathbf{K}} = \bigcup_{d \geq 1} \mathbf{K}((z^{1/d}))$ and $\mathscr{P}_{\overline{\mathbf{K}}} = \bigcup_{d \geq 1} \overline{\mathbf{K}}((z^{1/d}))$ denote the fields of Puiseux series with coefficients in $\mathbf{K}$ and $\overline{\mathbf{K}}$ respectively. Let $\mathscr{H}_{\overline{\mathbf{K}}} = \overline{\mathbf{K}}((z^{\mathbb{Q}}))$ denote the field of Hahn series with value group $\mathbb{Q}$ and coefficients in $\overline{\mathbf{K}}$, that is the set of formal series $\sum_{\gamma \in \mathbb{Q}} a_\gamma z^\gamma$ with $a_\gamma \in \overline{\mathbf{K}}$ such that $\{\gamma : a_\gamma \neq 0\}$ is a

[2] A basis of solutions of (2) is a set of m solutions which are linearly independent over the field of constants $\overline{\mathbf{K}}$.

[3] We assume that we are provided with a way to compute as many coefficients as needed in the power series expansion of $a_0, \dots, a_m$.

well-ordered subset of $\mathbb{Q}$ with respect to the restriction of the order $<$. The map ϕ_p extends to $\mathscr{P}_{\mathbf{K}}$, $\mathscr{P}_{\overline{\mathbf{K}}}$ and $\mathscr{H}_{\overline{\mathbf{K}}}$ in the obvious way. Roques and the first author [FR25, Theorem 17] proved that any p-Mahler system (6) over $\mathbf{K}((z))$ has a *fundamental matrix of solutions* of the form

$$PHe_C \tag{7}$$

where

(a) the matrix $P \in \mathrm{GL}_m(\mathscr{P}_{\overline{\mathbf{K}}})$ is such that $\Theta := \phi_p(P)^{-1}AP$ is block upper triangular with entries in $\overline{\mathbf{K}}[z^{-1/*}]$ and constant non-singular diagonal blocks;

(b) the matrix H
- belongs to $\mathrm{GL}_m(\mathscr{H}_{\overline{\mathbf{K}}})$ and satisfies $\phi_p(H)C = \Theta H$, where $C \in \mathrm{GL}_m(\overline{\mathbf{K}})$ is the constant term of Θ,
- is block upper triangular with blocks fitting with the block decomposition of Θ and diagonal blocks equal to the identity;
- has coefficients which are $\overline{\mathbf{K}}[z^{\pm 1/d}]$-linear combinations of some $\boldsymbol{\xi}_\omega$, $\boldsymbol{\omega} \in \Omega$, where $d \in \mathbb{Z}_{>0}$;

(c) the matrix e_C has coefficients which are $\overline{\mathbf{K}}$-linear combinations of the $e_c\ell^j$, $c \in \overline{\mathbf{K}}^\times$, $j \in \mathbb{Z}_{\geq 0}$, and satisfies $\phi_p(e_C) = Ce_c$.

We make the computation of such a fundamental matrix of solutions effective.

Theorem 2. *Let $\mathbf{K}$ be an effective field. Consider a p-Mahler system (6) over $\mathbf{K}((z))$. Algorithms 1, 2 and 3 provide a fundamental matrix of solutions of the form (7). More precisely:*

- *Algorithm 1 provides a solution to (a) over $\mathbf{K}$, that is a matrix with the required form $\Theta \in \mathrm{GL}_m(\mathbf{K}[z^{-1/*}])$ and a truncation up to any order of a matrix $P \in \mathrm{GL}_m(\mathscr{P}_{\mathbf{K}})$ of Puiseux series with coefficients in $\mathbf{K}$ such that $\Theta = \phi_p(P)^{-1}AP$;*
- *Algorithm 2 provides a solution to (b), that is a description of the entries of H as $\overline{\mathbf{K}}$-linear combinations of some $\boldsymbol{\xi}_{\boldsymbol{\omega}}$, $\boldsymbol{\omega} \in \Omega$;*
- *Algorithm 3 provides a solution to (c), that is a description of the entries of e_C as linear combinations of some $e_c\ell^j$, $c \in \overline{\mathbf{K}}^\times$, $j \in \mathbb{Z}_{\geq 0}$.*

Remark.
- It follows from Theorem 2 that a solution to (a) exists with $\mathbf{K}$ as a base field instead of $\overline{\mathbf{K}}$. Such a fact does not follow from [FR25] directly but is the consequence of the construction in Section 4. In contrast, it is not always possible to perform (c) on $\mathbf{K}$ (see Section 5.3).
- The entries of the matrix H produced by Algorithm 2 are linear combinations of some $\boldsymbol{\xi}_{\boldsymbol{\omega}}$ over the field $\overline{\mathbf{K}}$, rather than over the ring $\overline{\mathbf{K}}[z^{\pm 1/d}]$ as was the case in (b). This refines the results from [FR25].

To any linear equation (2) one can associate a system by considering its companion matrix. If one obtains a fundamental matrix of solutions of this system thanks to Theorem 2, then one gets a solution as in Theorem 1 by looking at the first row of PHe_C.

1.3. **Organization of the paper.** In Section 2 we discuss connections between Mahler equations and other areas of research. This section is independent of the remainder of the paper and may be skipped on a first reading. The main contribution of this paper is Algorithm 1 whose proof occupies the next two sections. In Section 3 we gather preliminary results about the matrices P and Θ introduced in Section 1.2 that will reduce the problem to a finite-dimensional one over $\mathbf{K}$. In Section 4, we present and

prove the correctness of Algorithm 1. Once one knows Θ, in Section 5 we describe Algorithms 2 and 3 which allow us to compute the matrices H and e_C, respectively, introduced in Section 1.2. They provide algorithmic versions of the constructions from [FR25] and [Roq18]. Then, we state Algorithm 4 which is the core of Theorem 1. Eventually, in Section 6, we run this algorithm on two examples.

2. Applications and connections

The study of Mahler equations is motivated by their occurrence in diverse areas of mathematics. The algorithms developed in this paper provide computational tools relevant to each of these domains. We outline some key connections in what follows. This material is not needed for the rest of the paper. Thus, the reader should feel free to proceed directly to Section 3.

Automatic sequences and automatic real numbers. Mahler equations naturally occur in the study of finite automata. A deterministic finite automaton can be used as a transducer to produce a sequence $(u_n)_{n\in\mathbb{N}}$ where u_n denotes the output of the automaton when it reads the expansion of n in base p (see [AS03] for details). When the elements u_n belong to some field $\mathbf{K}$, a classical result states that the power series $\sum_n u_n z^n$ satisfies some linear Mahler equation with coefficients in $\mathbf{K}[z]$ (see [Cob68]). When $\mathbf{K} := \mathbb{Q}$, combined with Mahler's method, this connection has been used to establish the following results[4]:

- the base-b expansion of an irrational algebraic real number cannot be produced by a finite automaton [Phi15, AF17];
- the expansions in two multiplicatively independent bases of an irrational real number cannot both be produced by automata [AF25].

It has also been used to reprove and generalize the well-known Cobham's theorem [SS19, AF25].

Regular sequences. Regular sequences are a generalization of sequences produced by a finite automaton. Strictly speaking, the p-regular sequences are the sequences that may produce a weighted finite automaton when reading the integers written in base p. Among them one finds the sequences whose nth term is the sum of digits of n in base p, the p-adic valuation of n or of $n!$, the complexity of the merge-sort algorithm in a set with n elements, the number of odd entries in the nth row of Pascal's triangle, the nth Cantor number. As established by Becker [Bec94], the generating series of a p-regular sequence is solution to some p-Mahler equation.

Algebraic power series in $\mathbb{F}_p[[z]]$. When p is prime, the elements of $\mathbb{F}_p[[z]]$ that are algebraic over $\mathbb{F}_p[z]$ are exactly the solutions of linear Mahler equations over $\mathbb{F}_p[z]$. Actually, given an algebraic equation over $\mathbb{F}_p[z]$, deriving a p-Mahler equation whose solutions in $\mathbb{F}_p[[z]]$ are the power series solutions of the former algebraic equation is straightforward, using the identity $f(z^p) = f(z)^p$ in $\mathbb{F}_p[[z]]$. For example, when $p = 2$, any solution to

$$c_0 + c_1 y(z) + c_2 y(z)^2 = 0,$$

with $c_0, c_1, c_2 \in \mathbb{F}_p[z]$ and $c_0 c_1 c_2 \neq 0$, satisfies the equation

$$c_1 c_0 y(z) + (c_1^2 - c_2 c_0) y(z)^2 - c_2^2 y(z)^4 = 0$$

[4]The first result was originally proved using a different approach, see [AB07].

which, for power series in $\mathbb{F}_2[[z]]$, is equivalent to the Mahler equation

$$c_1c_0y(z) + (c_1^2 - c_2c_0)y(z^2) - c_2^2y(z^{2^2}) = 0.$$

One should note, however, that this does not imply that our main algorithm for Mahler equations provides all solutions of the initial algebraic equation. Indeed, the identity $f(z^p) = f(z)^p$ does not extend to $\mathscr{H}_{\overline{\mathbb{F}_p}}[(e_c)_c, \ell]$.

Periods of Drinfeld modules. The theory of Drinfeld modules in positive characteristic provides analogs of numerous famous periods. As shown by Denis [Den00], some of them arise as values of solutions of Mahler equations. For example, a p-analog of π is given by $\prod_{n\geq 0}(1-\theta^{1-p^n})$ in the completion $\mathbf{K}$ of the algebraic closure of $\mathbb{F}_p((\theta^{-1}))$. This is the value at θ^{-1} of the function $f(z) := \prod_{n\geq 0}(1-\theta z^{p^n}) \in \mathbf{K}[[z]]$ which satisfies the p-Mahler equation

$$f(z) - (1-\theta z)f(z^p) = 0\,.$$

Another example is the analog ζ_C of the classical Riemann ζ function. Carlitz [Car35] proved that $\zeta_C(s) = f_s(\theta)$ for $s \in \{1, \ldots, p-1\}$, where $f_s(z) \in \mathbf{K}[[z]]$ satisfies the inhomogeneous p-Mahler equation

$$f_s(z^p) - (-1)^s(z^p-\theta)^s f_s(z) = (-1)^s(z^p-\theta)^s\,.$$

Using this equation, Dennis [Den06] was able to prove the algebraic independence of $\zeta_C(1), \ldots, \zeta_C(p-1)$ over $\mathbb{F}_p(\theta)$ (see also [Fer18]).

Purity results. When considering the minimal differential equation satisfied by some E-function, André [And00] proved that the functions $f(z)$ appearing in (1) are E-functions as well. Surprisingly, such a "purity theorem", which does not concern the values of these functions, implies the celebrated Siegel–Shidlovskii theorem on algebraic relations between values of E-functions at algebraic points and even some remarkable refinements due to Beukers [Beu06]. A purity theorem analogous to André's has recently been established in the framework of Mahler equations [FR25] and similarly shares a deep connection with results concerning the values of solutions of these equations [ABS23]. One instance of this result is the following: if (2) is the minimal Mahler equation satisfied by some p-regular power series, then it has a basis of solutions of the form (5) for which each of the series $f_{i,c,j,\boldsymbol{\omega}}$ is p-regular too.

Galois theory of Mahler equations. Recent results have established that any algebraic relation between values of solutions of Mahler equations at some nonzero algebraic point has a functional origin (see [AF24]). The study of functional relations is a delicate task and constitutes the subject of difference Galois theory. The Galois group of a Mahler system (6) is a linear algebraic group which encodes the relations between the entries of any fundamental matrix of solutions. From this point of view, being able to compute a fundamental matrix of solutions of a Mahler system as in Theorem 2 should be of some help to understand the algebraic relations between solutions of Mahler equations.

3. Properties of the Puiseux part of a fundamental matrix of solutions

We fix an integer $p \geq 2$, a field $\mathbf{K}$ and let $\overline{\mathbf{K}}$ denote an algebraic closure of $\mathbf{K}$. The general form (7) of a fundamental matrix of solutions leads us to look for pairs (P, Θ) satisfying Condition (a). This section is devoted to the study of such pairs.

Definition 3. We say that a pair of $m \times m$ matrices (P, Θ) is admissible with respect to (6) if the following holds:

- Θ is block upper triangular with entries in $\mathbf{K}[z^{-1/*}]$ and has non-singular constant diagonal blocks;
- $P \in \mathrm{GL}_m(\mathscr{P}_{\mathbf{K}})$ is such that

$$\phi_p(P)\Theta = AP\,. \tag{8}$$

3.1. **Equations, systems and modules.** We briefly recall the vocabulary of equations, systems and modules.

Let $\mathscr{H}_{\mathbf{K}}$ denote the field of Hahn series with coefficients in $\mathbf{K}$ and value group $\mathbb{Q}$. An element of $\mathscr{H}_{\mathbf{K}}$ is a formal series $f(z) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ with $f_\gamma \in \mathbf{K}$ such that the set $\{\gamma : f_\gamma \neq 0\}$ is a well-ordered subset of $\mathbb{Q}$ for the total order $<$. This guarantees that $\mathscr{H}_{\mathbf{K}}$ is a field extension of $\mathbf{K}((z))$ (see [Roq18] for details).

Given a subfield $\mathbb{F}$ of $\mathscr{H}_{\mathbf{K}}$, we say that two p-Mahler systems $\phi_p(Y) = AY$ and $\phi_p(Y) = BY$ are $\mathbb{F}$-equivalent if there exists a matrix $F \in \mathrm{GL}_m(\mathbb{F})$ such that $A = \phi_p(F)BF^{-1}$. When $\mathbf{K}$ is algebraically closed, by the *cyclic vector lemma*[5] [FP22], any p-Mahler system

$$\phi_p(Y) = AY, \quad A(z) \in \mathrm{GL}_m(\mathbb{F}) \tag{9}$$

is $\mathbb{F}$-equivalent to a p-Mahler system whose matrix is of the form

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_0 & \cdots & \cdots & \cdots & a_{m-1} \end{pmatrix} \tag{10}$$

where $a_0, \dots, a_{m-1} \in \mathbb{F}$, $a_0 a_{m-1} \neq 0$. A vector Y is solution to the p-Mahler system with matrix B if and only if $Y = {}^{\mathrm{t}}(y, \phi_p(y), \dots, \phi_p^{m-1}(y))$ for some y solution to the following p-Mahler equation

$$a_0 y + a_1\phi_p(y) + \cdots + a_{m-1}\phi_p^{m-1}(y) - \phi_p^m(y) = 0\,. \tag{11}$$

Consider the ring $\mathscr{D}_{\mathbb{F}} := \mathbb{F}\langle\Phi\rangle$ of non-commutative polynomials in the indeterminate Φ with the property that $\Phi f = \phi_p(f)\Phi$ for any $f \in \mathbb{F}$. To any companion matrix B as in (10) we associate the operator $L_B = a_0 + a_1\Phi + \cdots + a_{m-1}\Phi^{m-1} - \Phi^m$. Then (11) may be rewritten as $L_B(y) = 0$.

To any system (9), we can associate a $\mathscr{D}_{\mathbb{F}}$-module M_A of finite rank as follows: the underlying $\mathbb{F}$-vector space is $\mathbb{F}^m$ and Φ acts on $\mathbb{F}^m$ by

$$\forall \boldsymbol{v} \in \mathbb{F}^m, \quad \Phi(\boldsymbol{v}) = A^{-1}\phi_p(\boldsymbol{v})\,.$$

Conversely, to any $\mathscr{D}_{\mathbb{F}}$-module M of finite rank we can associate a Mahler system by choosing a basis. It is not difficult to prove that if $\phi_p(Y) = AY$ and $\phi_p(Y) = BY$ are $\mathbb{F}$-equivalent then the $\mathscr{D}_{\mathbb{F}}$-modules M_A and M_B are isomorphic. Furthermore, if B is a companion matrix, M_B is isomorphic to the module $\mathscr{D}_{\mathbb{F}}/\mathscr{D}_{\mathbb{F}}L_B$.

To any operator $L = a_0 + a_1\Phi + \cdots + a_m\Phi^m \in \mathscr{D}_{\mathbb{F}}$ we associate a *Newton polygon* $\mathcal{N}(L)$, which is the convex hull of the set of points $\{(p^i, j) : 0 \le i \le m,\ j \ge \operatorname{val} a_i\}$. We let $\mu_1, \dots, \mu_r \in \mathbb{Q}$ denote the finite slopes of $\mathcal{N}(L)$. Let $d(L)$ be the least integer

[5] The proof in [FP22] is written over $\overline{\mathbb{Q}}(z)$ but it works over $\mathbb{F}$ since its field of coefficients $\mathbf{K}$ is algebraically closed, hence, infinite.

d such that $\mu_1, \dots, \mu_r \in d^{-1}\mathbb{Z}[p^{-1}]$. This is the least integer d relatively prime with p and such that the denominators of $\mu_1, \dots, \mu_r$ divide dp^k for some k.

3.2. **Ramification order of an admissible pair.** Recall that we assume $\mathbf{K}$ to be algebraically closed. We now take $\mathbb{F} := \mathbf{K}((z))$ and consider a Mahler system

$$\phi_p(Y) = AY, \quad A(z) \in \mathrm{GL}_m(\mathbf{K}((z)))\,. \tag{12}$$

Then, the module M_A is isomorphic to some $\mathscr{D}_{\mathbf{K}((z))}$-module $\mathscr{D}_{\mathbf{K}((z))}/\mathscr{D}_{\mathbf{K}((z))}L$. We set $d(A) := d(L)$. It can be proved that this does not depend on the choices of L such that $M_A \simeq \mathscr{D}_{\mathbf{K}((z))}/\mathscr{D}_{\mathbf{K}((z))}L$, a fact we do not need here. Note that $d(A(z^{d(A)})) = 1$ for any matrix A, so we can always reduce to the case where $d(A) = 1$. Note also that $d(A) \mid p^m - 1$ so that $d(A(z^{p^m-1})) = 1$.

Proposition 4. *Suppose that* $\mathbf{K}$ *is algebraically closed. Then, the system* (12) *admits an admissible pair* (P, Θ) *such that* P *and* Θ *have entries in* $\mathbf{K}((z^{1/d(A)}))$.

As mentioned in the introduction, we will obtain as a by-product of Algorithm 1 that this proposition still holds when $\mathbf{K}$ is not algebraically closed.

Proof. Up to replacing z by $z^{d(A)}$, we may assume that $d(A) = 1$. Then, Proposition 4 can be reformulated as follows: *the system* (12) *is* $\mathbf{K}((z))$*-equivalent to a block upper triangular system with matrix in* $\mathrm{GL}_m(\mathbf{K}[z^{-1}])$ *and constant diagonal blocks.* Since we closely follow the proof of Step 1 in [FR25, Theorem 17], we shall be brief and present only the necessary adaptations. The existence of an admissible pair follows from a construction in two parts. The first one is a factorization of Mahler operators which implies that any Mahler system is Puiseux-equivalent to an upper triangular system with constant diagonal coefficients and Puiseux above-diagonal coefficients. The second part is a construction which implies that such a system is Puiseux-equivalent to a block upper triangular system with constant diagonal blocks and whose above-diagonal blocks have entries in $\mathbf{K}[z^{-1}]$. We only have to explain that these two steps may be carried out over $\mathbf{K}((z))$ instead of the field of Puiseux series and that $\mathbf{K}$ may be any algebraically closed field.

Part 1: Factorization of Mahler operators. We prove the following claim: *if* $L \in \mathscr{D}_{\mathbf{K}((z))}$ *is such that* $d(L) = 1$*, then one has* $L = L_1 \cdots L_m$ *with* $L_i \in \mathscr{D}_{\mathbf{K}((z))}$ *of degree* 1 *in* Φ.

The proof is an adaptation of the one of [Roq24, Proposition 15]. By induction, it is enough to prove that L admits a factorization $L = MN$, with $M, N \in \mathscr{D}_{\mathbf{K}((z))}$, N of degree 1 in Φ and $d(M) = 1$ (then, apply the induction hypothesis to M). The first slope μ_1 of $\mathcal{N}(L)$ is of the form $\mu_1 = a/(p^k - 1)$ for some $a \in \mathbb{Z}$ and $k \geq 1$. Since $d(L) = 1$, μ_1 has to be an integer. Without loss of generality, we may suppose that $\mu_1 = 0$. Indeed, with the notation of [Roq24], this amounts in replacing L with $L^{[\theta_{\mu_1}]}$, which does not modify $d(L)$ since $\mu_1 \in \mathbb{Z}$ ([Roq24, Lemma 17]). By [Roq24, Lemma 20], there exists $f \in \mathscr{H}_{\mathbf{K}}$ such that $f(0) = 1$ and $L(fe_c) = 0$ for some $c \in \mathbf{K}^\times$. The proof given there readily extends to any algebraically closed field $\mathbf{K}$. Since L has coefficients in $\mathbf{K}((z))$, f actually belongs to $\mathbf{K}((z))$ (see [FP25, Lemma 28]). Let M denote the result of the right Euclidian division of L by $N := (\Phi - c)f^{-1}$ in $\mathscr{D}_{\mathbf{K}((z))}$. Then $M \in \mathscr{D}_{\mathbf{K}((z))}$ and $L = MN$. By [Roq24, Lemma 21], each slope of M is p times a slope of L. *A fortiori*, $d(M) = 1$. This proves the claim.

Part 2: Reduction to $\mathbf{K}[z^{-1}]$. Let $L \in \mathscr{D}_{\mathbf{K}((z))}$ be such that $M_A \simeq \mathscr{D}_{\mathbf{K}((z))}/\mathscr{D}_{\mathbf{K}((z))}L$. From the first step, there exists $L_1, \dots, L_m \in \mathscr{D}_{\mathbf{K}((z))}$ of order 1 such that

$$L = L_1 \cdots L_m\,.$$

Arguing as in [FR25, Section 4.1.1], we conclude that the Mahler system (12) is $\mathbf{K}((z))$-equivalent to an upper triangular system with constant diagonal coefficients and above-diagonal coefficients in $\mathbf{K}((z))$. Then we follow the construction in [FR25, Section 4.1.2] without any modification. The key argument is that [FR25, Lemma 20] remains valid when one replaces the field of Puiseux series with $\mathbf{K}((z))$ and when $\mathbf{K}$ is any field. The remainder of the construction proceeds identically. □

3.3. **Valuations of an admissible pair.** The aim of this section is to bound from below $\operatorname{val} P$ and $\operatorname{val}\Theta$ for some admissible pair (P,Θ). We consider the following set

$$\mathcal{S}_p := \{s \in \mathbb{Z}_{<0} \,:\, p \nmid s\} \cup \{0\}$$

of non-positive integers either equal to 0 or not divisible by p.

Proposition 5. *Suppose that* $\mathbf{K}$ *is algebraically closed and consider a Mahler system* (12) *with* $d(A) = 1$. *Then, there exists an admissible pair* (P,Θ) *for which*

(i) $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ *and* Θ *has support in* $\mathcal{S}_p$*;*
(ii) *one has*

$$\operatorname{val} P \geq \frac{\operatorname{val} A}{p-1}, \quad \textit{and} \quad \operatorname{val}\Theta \geq \frac{pm \operatorname{val} A - p \operatorname{val}(\det A)}{p-1}.$$

Furthermore, each admissible pair satisfying (i) *also satisfies* (ii).

The proof is divided in two steps, corresponding respectively to (i) and (ii).

3.3.1. *Proof of* (i) *of Proposition 5.* By Proposition 4, the system has an admissible pair (P,Θ) with $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ and $\Theta \in \mathrm{GL}_m(\mathbf{K}[z^{-1}])$. Since $\mathbf{K}$ is algebraically closed, there exists such pair with Θ upper triangular. For such pairs we let $\theta_1,\dots,\theta_m$ denote the diagonal entries of Θ and $\theta_{i,j}(z)$, $i<j$, denote its above diagonal entries.

Let $\mathcal{I} := \{(i,j) \in \{1,\dots,m\}^2 \,:\, i<j\}$. We define an order on $\mathcal{I}$ as follows: $(i,j) \prec (k,l)$ if either $j<l$ or ($j=l$ and $i>k$). To scan the above-diagonal entries of $\Theta(z)$ with respect to this order one proceeds from left-to-right and then from bottom-to-top.

Suppose by contradiction that there exists no admissible pairs (P,Θ) for which $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ and Θ is uper triangular with support is in $\mathcal{S}_p$. Thus, for any admissible pair (P,Θ) with $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ and Θ upper triangular, there exists $(i,j) \in \mathcal{I}$ such that $\theta_{i,j}(z)$ has a non-zero multiple of p in its support: we let (i_Θ, j_Θ) be the least of them with respect to $\prec$. Consider the pairs (P,Θ) such that (i_Θ, j_Θ) takes its maximal value with respect to $\prec$ and let (i_0,j_0) denote this value. Recall that, for any (P,Θ) such that $(i_\Theta, j_\Theta) = (i_0,j_0)$, the support of $\Theta_{i,j}(z)$ for each $(i,j) \prec (i_0,j_0)$ is included in $\mathcal{S}_p$. Furthermore, by assumption, for any pair (P,Θ) such that $(i_\Theta, j_\Theta) = (i_0,j_0)$, there exists $\nu \in \mathbb{Z}_{>0}$ such that $z^{-\nu p}$ belongs to the support of $\theta_{i_0,j_0}(z) \in \mathbf{K}[z^{-1}]$. When the pair (P,Θ) is fixed, we let ν_Θ denote the greatest such ν. Then, we let ν_0 denote the minimum of the integers ν_Θ, among all the pairs (P,Θ) for which $(i_\Theta, j_\Theta) = (i_0,j_0)$. From now on we fix a pair (P,Θ) with $(i_\Theta, j_\Theta) = (i_0,j_0)$ and $\nu_\Theta = \nu_0$. We shall obtain a contradiction.

Let η denote the coefficient of $z^{-\nu_0 p}$ in $\theta_{i_0,j_0}(z)$. Let M denote the matrix which is the identity but for its (i_0,j_0)th entry which is equal to $\eta\theta_{j_0}^{-1} z^{-\nu_0}$. Set

$$\widetilde{\Theta} := \phi_p(M)^{-1}\Theta M.$$

This matrix is upper triangular, with diagonal coefficients $\theta_1, \dots, \theta_m$ and above diagonal coefficients in $\mathbf{K}[z^{-1}]$. Furthermore, when $(i,j) \prec (i_0, j_0)$ the (i,j)th coefficient of $\widetilde{\Theta}$ is the same as the one of Θ and the (i_0, j_0)th coefficient of $\widetilde{\Theta}$ is equal to

$$\widetilde{\theta}_{i_0,j_0}(z) := \theta_{i_0,j_0}(z) - \eta z^{-\nu_0 p} + \eta \theta_{i_0} \theta_{j_0}^{-1} z^{-\nu_0} . \tag{13}$$

Last, setting $\widetilde{P} = PM$ we have $\phi_p(\widetilde{P})\widetilde{\Theta} = A\widetilde{P}$ so that the pair $(\widetilde{P}, \widetilde{\Theta})$ is admissible. By maximality of (i_0, j_0), the support of $\widetilde{\theta}_{i_0,j_0}(z)$ has an element of the form $-\mu p$, for some $\mu \in \mathbb{Z}_{\geq 1}$. By minimality of ν_0, $\mu \geq \nu_0$. This contradicts (13). □

3.3.2. *Proof of* (ii) *of Proposition 5.* Let (P, Θ) be an admissible pair satisfying (i). Up to conjugation we may and will assume that Θ is upper triangular. We start by establishing the lower bound on $\operatorname{val} P$. Let $P_1, \dots, P_m$ enumerate the columns of P from left to right. We proceed by contradiction. Suppose that the lower bound for $\operatorname{val} P$ does not hold and let i be the least integer such that $\operatorname{val} P_i < \operatorname{val} A/(p-1)$. We let

$$\theta_1(z), \dots, \theta_{i-1}(z), \lambda_i, 0, \dots, 0$$

denote the entries of the ith column of $\Theta(z)$. Then, we infer from (8) that

$$\lambda_i \phi_p(P_i) = AP_i - \theta_1(z)\phi_p(P_1) - \cdots - \theta_{i-1}(z)\phi_p(P_{i-1})$$

Since $\operatorname{val} P_i < \operatorname{val} A/(p-1)$, we have $\operatorname{val}(AP_i) \geq \operatorname{val}(A) + \operatorname{val}(P_i) > p \operatorname{val}(P_i) = \operatorname{val}(\lambda_i \phi_p(P_i))$. Thus, the valuation of $\phi_p(P_i)$ is equal to the valuation of

$$\theta_1(z)\phi_p(P_1) + \cdots + \theta_{i-1}(z)\phi_p(P_{i-1}) .$$

Such a valuation is of the form $\nu + p\mu$, where ν is in the support of some $\theta_j(z)$, $1 \leq j < i$, and $\mu \geq \min_{j<i} \operatorname{val}(P_j)$. Since $\operatorname{val}(\phi_p(P_i)) = p \operatorname{val}(P_i)$, p must divide ν. Since $\nu \in \mathcal{S}_p$, we have $\nu = 0$. Thus, $\operatorname{val} P_i = \mu \geq \min_{j<i} \operatorname{val}(P_j) \geq \operatorname{val} A/(p-1)$, a contradiction.

To establish the bound for $\operatorname{val} \Theta$ we first bound $\operatorname{val} P^{-1}$ from below. Since Θ is upper triangular with invertible constant diagonal entries, $\det \Theta \in \mathbf{K}^\times$. Hence, $\operatorname{val} \det \Theta = 0$ and we infer from (8) that

$$\operatorname{val}(\det P) = \frac{\operatorname{val}(\det A)}{p-1} . \tag{14}$$

From the formula $P^{-1} = (\det P)^{-1}\, {}^{\mathrm{t}}\mathrm{Com}\, P$, we deduce that

$$\begin{aligned} \operatorname{val} P^{-1} &= \operatorname{val}(\operatorname{Com} P) - \operatorname{val}(\det P) \\ &\geq (m-1) \operatorname{val} P - \frac{\operatorname{val}(\det A)}{p-1} \geq \frac{(m-1)\operatorname{val} A - \operatorname{val}(\det A)}{p-1} . \end{aligned}$$

Last, since $\Theta = \phi_p(P^{-1})AP$, we have

$$\operatorname{val} \Theta \geq p \operatorname{val} P^{-1} + \operatorname{val} A + \operatorname{val} P \geq \frac{pm \operatorname{val} A - p \operatorname{val}(\det A)}{p-1}$$

□

3.4. **Laurent series expansion of P.** We fix a Mahler system (12) for which $d(A) = 1$. We do not necessarily assume that $\mathbf{K}$ is algebraically closed. Note that if (P, Θ) is admissible it remains so when considering the equation over $\overline{\mathbf{K}}((z))$. In particular, the conclusion of Proposition 5 remains true. In the remainder of the paper we shall take the following notation:

$$\begin{aligned} &\nu_\Theta := \min\left\{ v \geq \frac{pm \operatorname{val} A - p \operatorname{val}(\det A)}{p-1} \ : \ v \in \mathcal{S}_p \right\}, \qquad \nu_P := \left\lceil \frac{\operatorname{val} A}{p-1} \right\rceil, \\ &\nu := \min\{\nu_P,\, p\nu_P + \operatorname{val} A^{-1}\} + \nu_\Theta, \\ &\mu := \max\left\{ \left\lceil -\frac{\operatorname{val} A^{-1} + \nu_\Theta}{p-1} \right\rceil, \frac{\operatorname{val}\det(A)}{p-1} - (m-1)\nu_P \right\}. \end{aligned} \tag{15}$$

The integers ν_P and ν_Θ should be considered as lower bounds for the valuations of the components of an admissible pair (P, Θ). Since $\operatorname{val}(\det A) \geq m \operatorname{val} A$ we have $\nu_\Theta \leq 0$. Last, it follows from (14) that μ is an integer.

Given an admissible pair (P, Θ) –if such a pair exists– one can group the columns of P according to the block decomposition of Θ. The aim of this section is to establish recurrence formulas for the Laurent series expansion of these groups of columns.

3.4.1. *A recurrence formula for the blocks of columns.* Let $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ and let $\Theta \in \mathrm{GL}_m(\mathbf{K}[z^{-1}])$ be block upper triangular with blocks of sizes $b_1, \ldots, b_s$, respectively, support in $\mathcal{S}_p$, $\operatorname{val}(P) \geq \nu_P$ and $\operatorname{val}\Theta \geq \nu_\Theta$. We do not assume at this stage that (P, Θ) is an admissible pair. We partition the columns of P according to the block decomposition of Θ and denote by $Q_1, \ldots, Q_s$ the corresponding matrices. Precisely, Q_j is the $m \times b_j$ matrix consisting of the columns $b_1 + \cdots + b_{j-1} + 1$ to $b_1 + \cdots + b_j$ of P. For $i < j$ we let $\Theta_{i,j}(z)$ denote the (i,j)-th block of Θ, which is a $b_i \times b_j$ matrix and $\Theta_1, \ldots, \Theta_s$ denote its diagonal blocks. We infer from Definition 3 that the pair (P, Θ) is admissible if and only if $\phi_p(P)\Theta = AP$, that is, if and only if

$$\forall j \in \{1, \ldots, s\},\ Q_j(z) = A^{-1}(z) Q_j(z^p)\Theta_j + \sum_{i=1}^{j-1} A^{-1}(z) Q_i(z^p)\Theta_{i,j}(z)\,. \tag{16}$$

By assumption, we may write

$$Q_j(z) =: \sum_{k \geq \nu_P} Q_{j,k} z^k, \quad \Theta_{i,j}(z) =: \sum_{k=\nu_\Theta}^{0} \Theta_{i,j,k} z^k, \quad A^{-1}(z) =: \sum_{k \geq \operatorname{val} A^{-1}} B_k z^k\,,$$

where $\Theta_{i,j,k} := 0$ when $k \notin \mathcal{S}_p$. Then, (16) is equivalent to: $\forall j \leq s$, $\forall n \geq \nu$,

$$Q_{j,n} = \sum_{k=\operatorname{val} A^{-1}}^{n-p\nu_P} B_k Q_{j,\frac{n-k}{p}} \Theta_j + \sum_{i=1}^{j-1} \sum_{k=\operatorname{val} A^{-1}}^{n-p\nu_P-\nu_\Theta} \sum_{l=\nu_\Theta}^{0} B_k Q_{i,\frac{n-k-l}{p}} \Theta_{i,j,l}\,. \tag{17}$$

Remark 6. Since $-\operatorname{val} A^{-1} - \nu_\Theta \leq (p-1)\mu$, if n is an integer such that $\nu \leq n \leq \mu$ (respectively $n > \mu$) then $\frac{n - \operatorname{val} A^{-1} - \nu_\Theta}{p} \leq \mu$ (respectively $< n$). Thus, for any $n \in \{\nu, \ldots, \mu\}$ the matrices $Q_{i,t}$ and $Q_{j,t}$ appearing in the right-hand side of (17) are such that $t \leq \mu$. Furthermore, if $n > \mu$, the matrices $Q_{i,t}$ and $Q_{j,t}$ appearing in the right-hand side of (17) are such that $t < n$. Thus, once the matrices $Q_{j,n}$, $n \in \{\nu, \ldots, \mu\}$, Θ_j and $\Theta_{i,j,k}$ are known, the matrices $Q_{j,n}$ with $n > \mu$ can be computed inductively.

3.4.2. *The window* $\{\nu, \dots, \mu\}$. We consider the map $\pi : \mathbf{K}((z))^m \to \mathbf{K}^{m(\mu-\nu+1)}$ defined by

$$\pi : \boldsymbol{f}(z) = \sum_n \boldsymbol{f}_n z^n \in \mathbf{K}((z))^m \mapsto {}^{\mathrm{t}}({}^{\mathrm{t}}\boldsymbol{f}_\nu, \dots {}^{\mathrm{t}}\boldsymbol{f}_\mu)\,.$$

Concretely, π maps any vector of Laurent series with size m to the vector obtained by concatenating the coefficients corresponding to $z^\nu, z^{\nu+1}, \dots, z^\mu$. Let F be a matrix whose columns F_i are vectors of Laurent series of size m. By abuse of notation we denote $\pi(F)$ the matrix whose columns are the $\pi(F_i)$. We let V_0 denote the vector space $\{0\}^{m(\nu_P-\nu)} \times \mathbf{K}^{m(\mu-\nu_P+1)}$, that is the image under π of the space of vectors $\boldsymbol{f} \in \mathbf{K}((z))^m$ for which $\operatorname{val} \boldsymbol{f} \geq \nu_P$. In particular, if (P, Θ) is an admissible pair as in Proposition 5, and $P_1, \dots, P_m$ are the columns of P, we have $\pi(P_i) \in V_0$. For each non-positive integer $l \in \{\nu_\Theta, \dots, 0\} \cap \mathcal{S}_p =: \mathcal{S}'_p$ we let M_l denote the square matrix defined by

$$M_l \pi(\boldsymbol{f}(z)) = \pi(z^l A^{-1}(z) \boldsymbol{f}(z^p)) \tag{18}$$

for all $\boldsymbol{f} \in \mathbf{K}((z))^m$ for which $\operatorname{val} \boldsymbol{f} \geq \nu_P$, and which is null on the supplementary space $\mathbf{K}^{m(\nu_P-\nu)} \times \{0\}^{m(\mu-\nu_P+1)}$ of V_0. From Remark 6, such matrices are well-defined. Note, for computational considerations, that these matrices M_l admit decompositions into blocks of $m \times m$ matrices denoted by $M_{l;i,j}$, $1 \leq i, j \leq \mu - \nu + 1$, such that:

$$M_{l;i,j} = \begin{cases} 0 & \text{if } j \in \{1, \dots, \nu_P - \nu\} \\ B_{i+\nu-l-1-p(j+\nu-1)} & \text{otherwise.} \end{cases}$$

3.4.3. *Reduction to a finite dimensional problem.* From now on, we let $M := M_0$. The map π and the matrices $M = M_0, M_{-1}, \dots, M_{\nu_\Theta}$ allow us to express the admissibility of a pair (P, Θ). Indeed, the equation (17) for $n \in \{\nu, \dots, \mu\}$ is equivalent to

$$\forall j \in \{1, \dots, r\}, \quad \pi(Q_j) = M\pi(Q_j)\Theta_j + \sum_{i<j} \sum_{l \in \mathcal{S}'_p} M_l \pi(Q_i) \Theta_{i,j,l}\,. \tag{19}$$

Conversely, we have the following.

Proposition 7. *Let $r \geq 1$ and let $b_1, \dots, b_r$ be integers with $b_1 + \dots + b_r = m$. For each $j \in \{1, \dots, r\}$, let E_j be a $m(\mu-\nu+1) \times b_j$ matrix. Suppose that the m columns of $E_1, \dots, E_r$ are $\mathbf{K}$-linearly independent and belong to V_0. Let $\Theta_1, \dots, \Theta_r$, $\Theta_{i,j,l}$, $l \in \mathcal{S}'_p$, be matrices with entries in $\mathbf{K}$, such that*

$$\forall j \in \{1, \dots, r\}, \quad E_j = ME_j\Theta_j + \sum_{i<j} \sum_{l \in \mathcal{S}'_p} M_l E_j \Theta_{i,j,l}\,. \tag{20}$$

Let Θ denote the $m \times m$ matrix with diagonal blocks $\Theta_1, \dots, \Theta_r$ and upper diagonal blocks $\Theta_{i,j}(z) = \sum_{l \in \mathcal{S}'_p} \Theta_{i,j,l} z^l$. Then, there exists a matrix $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ such that the pair (P, Θ) is admissible for (23) and $\pi(P) = (E_1 | \cdots | E_r)$.

We establish Proposition 7 after the following lemma.

Lemma 8. *Let P be an $m \times m$ matrix with entries in $\mathbf{K}((z))$. Let $\Theta \in \mathrm{GL}_r(\mathbf{K}[z^{-1}])$ be block upper triangular with constant diagonal blocks and support in $\mathcal{S}_p$. Suppose that $\phi_p(P)\Theta = AP$. If the columns of P are linearly independent over $\mathbf{K}$, then they are linearly independent over $\mathbf{K}((z))$, that is $P \in \mathrm{GL}_m(\mathbf{K}((z)))$, or, equivalently, (P, Θ) is an admissible pair.*

Proof. Let $P_1, \dots, P_m$ denote the columns of P. Since the entries of P are Laurent series with coefficients in $\mathbf{K}$, and $P_1, \dots, P_m$ are $\mathbf{K}$-linearly independent, they are $\overline{\mathbf{K}}$-linearly independent. Thus, up to enlarge $\mathbf{K}$, we may suppose that Θ is upper triangular with constant diagonal entries $\theta_1, \dots, \theta_m$. We argue by contradiction. Assume on the contrary that $P_1, \dots, P_m$ are $\mathbf{K}((z))$-linearly *dependent*. Let r be minimal such that $P_1, \dots, P_r$ are $\mathbf{K}((z))$-linearly dependent. Let $\boldsymbol{\lambda}(z) = {}^{\mathrm{t}}(\lambda_1(z), \dots, \lambda_r(z), 0, \dots, 0) \in \mathbf{K}((z))^m \setminus \{0\}$ be such that

$$\lambda_1 P_1 + \cdots + \lambda_r P_r = 0\,.$$

By minimality of r, $\lambda_r \neq 0$. We may then suppose that $\lambda_r = 1$. Since $P_1, \dots, P_r$ are $\mathbf{K}$-linearly independent, there exists an index $i \in \{1, \dots, r-1\}$ such that $\lambda_i \notin \mathbf{K}$. We let i_0 be the greatest of such indices. From the identity

$$\phi_p(P)\Theta\boldsymbol{\lambda} = AP\boldsymbol{\lambda} = 0,$$

the fact that Θ is upper triangular and the fact that r is minimal, we deduce that there exists $h(z) \in \mathbf{K}((z))$ such that

$$\Theta(z)\boldsymbol{\lambda}(z) = h(z)\boldsymbol{\lambda}(z^p).$$

Looking at the rth coordinate on both side, we get $h(z) = \theta_r$. Thus,

$$\Theta(z)\boldsymbol{\lambda}(z) = \theta_r\boldsymbol{\lambda}(z^p).$$

Let $(0, \dots, 0, \theta_{i_0}, t_{i_0+1}(z), \dots, t_m(z))$, $t_i \in \mathbf{K}[z^{-1}]$, denote the i_0th row of Θ. Then,

$$(0, \dots, 0, \theta_{i_0}, t_{i_0+1}(z), \dots, t_m(z))\boldsymbol{\lambda}(z) = \theta_r\lambda_{i_0}(z^p)$$

thus

$$\theta_{i_0}\lambda_{i_0}(z) + t(z) = \theta_r\lambda_{i_0}(z^p) \tag{21}$$

where $t(z) \in \mathbf{K}[z^{-1}]$ has support in $\mathcal{S}_p$. Looking at the valuation on both sides, one of the following mutually exclusive situations holds:

(a) $\operatorname{val}\lambda_{i_0} = \operatorname{val} t < p \operatorname{val}\lambda_{i_0}$,
(b) $\operatorname{val}\lambda_{i_0} = p\operatorname{val}\lambda_{i_0} < \operatorname{val} t$,
(c) $\operatorname{val} t = p\operatorname{val}\lambda_{i_0}$.

The first two situations are impossible since $\operatorname{val} t \leq 0$. Thus (c) holds and $\operatorname{val}\lambda_{i_0} = \frac{\operatorname{val} t}{p}$. Since $\operatorname{val}\lambda_{i_0}$ is an integer and $\operatorname{val} t \in \mathcal{S}_p$, we have $\operatorname{val} t = 0$ that is $t \in \mathbf{K}$ and $\operatorname{val}\lambda_{i_0} = 0$. Then, Equation (21) forces λ_{i_0} to be in $\mathbf{K}$, a contradiction. □

Proof of Proposition 7. For $j \in \{1, \dots, r\}$, we define $m \times b_j$ matrices of Laurent series $Q_j = \sum_{n \geq \nu_P} Q_{j,n} z^n$ in the following way:

- $Q_{j,\nu_P}, \dots, Q_{j,\mu}$ are uniquely defined by the identity $\pi(Q_j) = E_j$.
- Suppose that $n \geq \mu$ and that $Q_{i,k}$ is defined for any $i \leq j$ and $k < n$. Then $Q_{j,n}$ is uniquely defined by (17).

By (19) and (20), $Q_1, \dots, Q_r, \Theta$ satisfy (17). Let $P = (Q_1|\cdots|Q_r)$, so that $\phi_p(P)\Theta = AP$. By construction, $\pi(P) = (E_1|\cdots|E_r)$. Thus, it only remains to prove that the columns of P are $\mathbf{K}((z))$-linearly independent. Since the columns of $E_1, \dots, E_r$ are $\mathbf{K}$-linearly independent, the columns of P are $\mathbf{K}$-linearly independent. $P_1, \dots, P_r$. Then, the result follows from Lemma 8. □

3.5. **Independence of the columns' projections.** We continue with $\mathbf{K}$ a field that we do not assume to be algebraically closed. Proposition 7 implies that we only need to compute $\pi(P)$ and Θ in order to obtain any coefficient of the matrix P of an admissible pair (P, Θ). The following lemma guarantees that the matrix $\pi(P)$ has maximal rank.

Lemma 9. *Let (P, Θ) be an admissible pair for (12) with $P \in \mathrm{GL}_m(\mathbf{K}((z))$ and Θ whose support is in $\mathcal{S}_p$. Then, the m columns of $\pi(P)$ are $\mathbf{K}$-linearly independent.*

Proof. Let $P_1, \dots, P_m$ denote the columns of P. Suppose on the contrary that the column vectors $\pi(P_1), \dots, \pi(P_m)$ are $\mathbf{K}$-linearly dependent and let $\lambda_1, \dots, \lambda_m \in \mathbf{K}$, not all zero, be such that

$$\lambda_1 \pi(P_1) + \cdots + \lambda_m \pi(P_m) = 0\,.$$

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$. Then, by definition of π and μ,

$$\mathrm{val}(P\boldsymbol{\lambda}) > \mu \geq \frac{\mathrm{val}\det(A)}{p-1} - (m-1)\nu_P.$$

Let i_0 be such that $\lambda_{i_0} \neq 0$ and let Λ denote the matrix which is the identity but for the i_0th column which is equal to $\boldsymbol{\lambda}$. Then, the matrix $P\Lambda$ has $m-1$ columns with valuation at least ν_P and one column with valuation at least $\frac{\mathrm{val}\det(A)}{p-1} - (m-1)\nu_P$. Thus,

$$\mathrm{val}\det(P\Lambda) > (m-1)\nu_P + \frac{\mathrm{val}\det(A)}{p-1} - (m-1)\nu_P = \frac{\mathrm{val}\det(A)}{p-1}\,.$$

Meanwhile, $\mathrm{val}\det(P\Lambda) = \mathrm{val}\det(P)$, which is equal to $\frac{\mathrm{val}\det(A)}{p-1}$ from (14). This is a contradiction. □

3.6. **A naive approach that we will not pursue.** Let $\mathbf{K}$ denote an effective field. A naive approach for computing (P, Θ) would consist in computing the columns of P inductively. Indeed, the algebraic closure $\overline{\mathbf{K}}$ of $\mathbf{K}$ remains effective, so that we could assume $\mathbf{K}$ to be algebraically closed. Then, it is sufficient to find a pair for which Θ is upper triangular. Then, (20) with $r = m$ and $b_1 = \cdots = b_m = 1$ provides an equation for the jth column of $\pi(P)$ if one knows the previous columns. Precisely, we must look for a pair $(\boldsymbol{v}, \lambda) \in V_0 \times \mathbf{K}^\times$ such that

$$(\mathrm{I} - \lambda M)\boldsymbol{v} \in \mathfrak{E}_j \tag{22}$$

where $\mathfrak{E}_j$ is the vector space spanned by the vectors $M_l E_i$, $l \in \mathcal{S}'_p$, $1 \leq i \leq j-1$, and $E_1, \dots, E_{j-1}$ are the first columns of $\pi(P)$ that we already computed. Furthermore, we know that λ is one of the diagonal entries of Θ, which means that it is an exponent of the system, following the terminology introduced in [Roq18]. Since the set of exponents of (12) is computable and finite, solving Equation (22) reduces to an elementary problem of linear algebra. One may compute this way m columns $E_1, \dots, E_m$ satisfying (20) and obtain this way an admissible pair (P, Θ) as in Proposition 7.

However, this method has several drawbacks compared to the algorithm we present in the next section. In particular, it unnecessarily requires working over the algebraic closure of $\mathbf{K}$.

4. Computing the Puiseux part of a fundamental matrix of solutions

The approach described in Section 3.6 requires working over an algebraic closure of $\mathbf{K}$. This can be avoided as we shall now see. We let $\mathbf{K}$ be an effective field that we do not assume to be algebraically closed. We consider a Mahler system

$$\phi_p(Y) = AY, \quad A(z) \in \mathrm{GL}_m(\mathbf{K}((z))) \tag{23}$$

for which $d(A) = 1$. Proposition 4 does not allow us to conclude that there exists a pair (P, Θ) admissible with respect to (23), with $P \in \mathrm{GL}_m(\mathbf{K}((z)))$, since $\mathbf{K}$ is not necessarily algebraically closed. This fact will follow from our construction. Recall that if there exists an admissible pair (P, Θ) and Θ has its support in $\mathcal{S}_p$, then the vector space spanned by the columns of $\pi(P)$ has dimension m and is included in

$$V_0 := \{0\}^{m(\nu_P - \nu)} \times \mathbf{K}^{m(\mu - \nu_P + 1)}$$

since $\mathrm{val}(P) \geq \nu_P$. According to the block structure of Θ, one can decompose P as $(Q_1|\cdots|Q_r)$ where Q_i are matrices with m rows each. Our construction consists of computing a sequence of vector spaces $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \cdots \subset V_0$ with the property that $\pi(Q_i) \subset \mathfrak{X}_i$ for each i, for any such admissible pair (P, Θ). We establish that one of these vector spaces has dimension m at some point. By choosing an appropriate basis, we are able to build an admissible pair.

4.1. **Construction of the vector spaces.** Recall that $\mathcal{S}'_p := \mathcal{S}_p \cap \{\nu_\Theta, \ldots, 0\}$. We define a sequence of vector spaces $(\mathfrak{X}_j)_{j \geq 0}$ by induction on j as follows.

- We let $\mathfrak{X}_0 := \{0\}$.
- Let $j \in \mathbb{Z}_{\geq 0}$ and assume that $\mathfrak{X}_j$ has been defined. We let $\mathfrak{U}_j := \mathrm{span}_{\mathbf{K}}(M_k \mathfrak{X}_j : k \in \mathcal{S}'_p)$ and $\mathfrak{X}_{j+1}$ be the largest subspace of V_0 for which

$$M\mathfrak{X}_{j+1} \subset \mathfrak{X}_{j+1} + \mathfrak{U}_j \quad \text{and} \quad \mathfrak{X}_{j+1} \subset M\mathfrak{X}_{j+1} + \mathfrak{U}_j. \tag{24}$$

Notice that $\mathfrak{U}_0 = \{0\}$ and $\mathfrak{X}_1$ is the largest subspace of V_0 such that $M\mathfrak{X}_1 = \mathfrak{X}_1$.

Lemma 10. *The sequences $(\mathfrak{X}_j)_{j \geq 0}$ and $(\mathfrak{U}_j)_{j \geq 0}$ are non-decreasing and, for all $j \in \mathbb{Z}_{\geq 0}$, $\mathfrak{X}_j \subset \mathfrak{U}_j$.*

Proof. We prove by induction on $j \geq 0$ that $\mathfrak{X}_j \subset \mathfrak{U}_j$, $\mathfrak{X}_j \subset \mathfrak{X}_{j+1}$ and $\mathfrak{U}_j \subset \mathfrak{U}_{j+1}$. When $j = 0$ the result is clear. Let $j \geq 1$ and assume the property true with $j - 1$. By definition, $\mathfrak{X}_j \subset M\mathfrak{X}_j + \mathfrak{U}_{j-1}$. Since $M = M_0$ and $0 \in \mathcal{S}'_p$, we have $M\mathfrak{X}_j \subset \mathfrak{U}_j$. Moreover, by induction hypothesis, $\mathfrak{U}_{j-1} \subset \mathfrak{U}_j$. Thus, $\mathfrak{X}_j \subset \mathfrak{U}_j$. Since $\mathfrak{X}_j \subset V_0$, to prove that $\mathfrak{X}_j \subset \mathfrak{X}_{j+1}$ we just have to prove that $\mathfrak{X}_j$ satisfies (24). This follows from the fact that $M\mathfrak{X}_j \subset \mathfrak{U}_j$ and $\mathfrak{X}_j \subset \mathfrak{U}_j$. The inclusion $\mathfrak{U}_j \subset \mathfrak{U}_{j+1}$ is an immediate consequence of the fact that $\mathfrak{X}_j \subset \mathfrak{X}_{j+1}$. □

Before going to our main result, we provide a method to compute the spaces $\mathfrak{X}_j$. Consider the non-increasing sequence $(\mathfrak{F}_{j,\ell})_{\ell \geq 0}$ of vector spaces defined by $\mathfrak{F}_{j,0} := V_0$ and, for any integer $\ell \geq 0$,

$$\mathfrak{F}_{j,\ell+1} := \mathfrak{F}_{j,\ell} \cap M^{-1}(\mathfrak{F}_{j,\ell} + \mathfrak{U}_j) \cap (M\mathfrak{F}_{j,\ell} + \mathfrak{U}_j)\,.$$

Lemma 11. *Let $j \geq 0$ and let ℓ_j be the least integer for which $\mathfrak{F}_{j,\ell_j} = \mathfrak{F}_{j,\ell_j+1}$. Then $\mathfrak{X}_{j+1} = \mathfrak{F}_{j,\ell_j}$ and $\ell_j \leq m(\mu - \nu_P + 1)$.*

Proof. First, we prove by induction on $\ell \geq 0$ that $\mathfrak{X}_{j+1} \subset \mathfrak{F}_{j,\ell}$. By assumption, $\mathfrak{X}_{j+1} \subset \mathfrak{F}_{j,0} = V_0$. Let $\ell \geq 0$ and assume $\mathfrak{X}_{j+1} \subset \mathfrak{F}_{j,\ell}$. Then, it follows from (24) that $\mathfrak{X}_{j+1} \subset \mathfrak{F}_{j,\ell+1}$, which ends the induction.

The sequence $(\mathfrak{F}_{j,\ell})_\ell$ is a non-increasing sequence of sub-vector spaces of V_0. Since V_0 is finite dimensional, there exists a non-negative integer ℓ such that $\mathfrak{F}_{j,\ell} = \mathfrak{F}_{j,\ell+1}$. Let ℓ_j denote the least of all such integers. Then, $\mathfrak{F}_{j,\ell_j} \subset M^{-1}(\mathfrak{F}_{j,\ell_j} + \mathfrak{U}_j) \cap (M\mathfrak{F}_{j,\ell_j} + \mathfrak{U}_j)$ which implies that $\mathfrak{F}_{j,\ell_j}$ satisfies (24). Since $\mathfrak{F}_{j,\ell_j} \subset V_0$, by maximality of $\mathfrak{X}_{j+1}$, we have $\mathfrak{X}_{j+1} \supset \mathfrak{F}_{j,\ell_j}$. Then, the beginning of the proof implies that $\mathfrak{X}_{j+1} = \mathfrak{F}_{j,\ell_j}$. Since $\dim(\mathfrak{F}_{j,0}) = \dim V_0 = m(\mu - \nu_P + 1)$, we have $\ell_j \leq m(\mu - \nu_P + 1)$. □

Remark. In [FP22], we were computing fundamental matrices of solutions in the case where the system is regular singular at 0. This corresponds precisely to the case where the matrix Θ may be taken to be constant. In other words, this corresponds to the case where the vector space $\mathfrak{X}_1$ has dimension m. Actually, in [FP22] our vector space, called $\mathfrak{X}_d$, was the greatest space to satisfy $M_d\mathfrak{X}_d = \mathfrak{X}_d$ and $\mathfrak{X}_d \subset \ker N_d$ for some explicit linear maps M_d and N_d. In the case $d = 1$, this vector space is closely related to the vector space $\mathfrak{X}_1$ defined here. Precisely, the vector space $\mathfrak{X}_d$ from [FP22] is the projection of the vector space $\mathfrak{X}_1$ onto the space where only the coefficients between ν_P and $\lceil -\operatorname{val} A^{-1}/(p-1)\rceil$ are taken into account. The map M_d corresponds to the restriction of the map M introduced here to this vector space, and the matrix N_d corresponds to the projection of this map onto its coefficients between $p\nu_P + \operatorname{val} A^{-1}$ and $\nu_P - 1$, whenever $p\nu_P + \operatorname{val} A^{-1} \leq \nu_P - 1$.

4.2. **Construction of an admissible pair.** Algorithm 1 relies on the following key-lemma.

Lemma 12. *We have* $\dim \mathfrak{X}_r \geq m$ *for some* $r \geq 0$. *For such an* r, *there exists a basis* $\boldsymbol{e}_{1,1}, \dots, \boldsymbol{e}_{1,m_1}, \boldsymbol{e}_{2,1}, \dots, \boldsymbol{e}_{r,m_r}$ *of* $\mathfrak{X}_r$ *fitting with the nested sequence* $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \cdots \mathfrak{X}_r$, *such that the following holds: letting* E_j, $j \in \{1, \dots, r\}$ *denote the matrix with columns* $\boldsymbol{e}_{j,1}, \dots, \boldsymbol{e}_{j,m_j}$, *there exists* $\Theta_j \in \mathrm{GL}_{m_j}(\mathbf{K})$ *such that*

$$E_j - ME_j\Theta_j \in \mathfrak{U}_{j-1}\,. \tag{25}$$

Remark. Using (25), we can write for each $j \geq 2$,

$$E_j - ME_j\Theta_j = \sum_{i<j} \sum_{k \in \mathcal{S}'_p} M_k E_i \Theta_{i,j,k}$$

for some matrices $\Theta_{i,j,k}$ with coefficients in $\mathbf{K}$. Then, from Proposition 7, there is an admissible pair (P, Θ) with $\pi(P) = (E_1|\cdots|E_r)$. In particular, $\dim \mathfrak{X}_r = m$.

Proof of Lemma 12. We start with the first statement. Let $\overline{\mathbf{K}}$ denote the algebraic closure of $\mathbf{K}$ and $V_0(\overline{\mathbf{K}})$, $\mathfrak{X}_j(\overline{\mathbf{K}})$, $\mathfrak{U}_j(\overline{\mathbf{K}})$ denote the $\overline{\mathbf{K}}$-vector spaces spanned by V_0, $\mathfrak{X}_j$ and $\mathfrak{U}_j$ respectively. Note that $\dim_{\overline{\mathbf{K}}} \mathfrak{X}_j(\overline{\mathbf{K}}) = \dim_{\mathbf{K}} \mathfrak{X}_j$. For any $j \in \{1, \dots, r\}$, we claim that $\mathfrak{X}_j(\overline{\mathbf{K}})$ is the greatest $\overline{\mathbf{K}}$-vector space $\mathfrak{X}$ for which $M\mathfrak{X} \subset \mathfrak{X} + \mathfrak{U}_{j-1}(\overline{\mathbf{K}})$ and $\mathfrak{X} \subset M\mathfrak{X} + \mathfrak{U}_{j-1}(\overline{\mathbf{K}})$. Indeed, since M has coefficients in $\mathbf{K}$, such equations are defined over $\mathbf{K}$ and the result follows[6] from the definition of $\mathfrak{X}_j$.

Let (P, Θ) be admissible for the system over $\overline{\mathbf{K}}((z))$, with $P \in \mathrm{GL}_m(\overline{\mathbf{K}}((z))$. Such a pair exists thanks to Proposition 4. Let $P = (Q_1|\cdots|Q_r)$ denote the decomposition of P fitting with the block decomposition of Θ. Let $E_j := \pi(Q_j)$ and let $\mathfrak{E}_j$ be the $\overline{\mathbf{K}}$-vector space spanned by its columns. By Lemma 9, the dimension of $\mathfrak{E}_1 + \cdots + \mathfrak{E}_r$ is equal to m. We are going to prove by induction on $j \geq 1$ that $\mathfrak{E}_j \subset \mathfrak{X}_j(\overline{\mathbf{K}})$. When

[6]This also follows by applying Lemma 11 twice, once with $\mathbf{K}$ and once with $\overline{\mathbf{K}}$ as a base field, and noticing that the vector spaces $\mathfrak{F}_{j,\ell}$ built in the latter case are obtained from the ones built in the former case by extension of the scalars from $\mathbf{K}$ to $\overline{\mathbf{K}}$.

$j=1$, (17) implies that $E_1 = ME_1\Theta_1$, which implies that $M\mathfrak{E}_1 = \mathfrak{E}_1$. By maximality of $\mathfrak{X}_1(\overline{\mathbf{K}})$, $\mathfrak{E}_1 \subset \mathfrak{X}_1(\overline{\mathbf{K}})$. Let $j \geq 2$ and suppose that $\mathfrak{E}_i \subset \mathfrak{X}_i(\overline{\mathbf{K}})$ for every $i < j$. From (17) we deduce that

$$E_j = ME_j\Theta_j + \sum_{i<j} \sum_{l=\nu_\Theta}^{0} M_l E_i \Theta_{i,j,l}\,.$$

Since $\mathfrak{E}_i \subset \mathfrak{X}_i(\overline{\mathbf{K}})$ for every $i < j$, the column vectors of the second term in the right-hand side belong to $\mathfrak{U}_{j-1}(\overline{\mathbf{K}})$. Since Θ_j is invertible, we get that

$$M\mathfrak{E}_j \subset \mathfrak{E}_j + \mathfrak{U}_{j-1}(\overline{\mathbf{K}}) \text{ and } \mathfrak{E}_j \subset M\mathfrak{E}_j + \mathfrak{U}_{j-1}(\overline{\mathbf{K}})\,.$$

By maximality of $\mathfrak{X}_j(\overline{\mathbf{K}})$, $\mathfrak{E}_j \subset \mathfrak{X}_j(\overline{\mathbf{K}})$. This ends the induction. Now, since $\mathfrak{X}_i \subset \mathfrak{X}_j$ when $i \leq j$, we have $\mathfrak{E}_1 + \cdots + \mathfrak{E}_r \subset \mathfrak{X}_r(\overline{\mathbf{K}})$. Thus,

$$\dim_{\mathbf{K}} \mathfrak{X}_r = \dim_{\overline{\mathbf{K}}} \mathfrak{X}_r(\overline{\mathbf{K}}) \geq \dim_{\overline{\mathbf{K}}} \mathfrak{E}_1 + \cdots + \mathfrak{E}_r = m.$$

Let us now prove that we can find a basis of $\mathfrak{X}_r$ as mentioned in the lemma. We only have to prove by induction on j that, for each $j \geq 1$, we can find $\boldsymbol{e}_{j,1}, \ldots, \boldsymbol{e}_{j,m_j}$ with the desired properties.

Let $j = 1$. Then, any basis $\boldsymbol{e}_{1,1}, \ldots, \boldsymbol{e}_{1,m_1}$ of $\mathfrak{X}_1$ has the required property, since in that case (25) reads $E_1 = ME_1\Theta_1$, which is true since $\mathfrak{X}_1 = M\mathfrak{X}_1$.

Suppose $j \geq 2$ and that we have built $\boldsymbol{e}_{1,1}, \ldots, \boldsymbol{e}_{j-1,m_{j-1}}$. Since $\mathfrak{X}_{j-1} \subset \mathfrak{U}_{j-1} \cap \mathfrak{X}_j$ by Lemma 10, we may decompose $\mathfrak{X}_j$ as follows

$$\mathfrak{X}_j = \mathfrak{X}_{j-1} \oplus \mathfrak{Y} \oplus \mathfrak{Z}$$

where $\mathfrak{Y}$ is any supplementary space of $\mathfrak{X}_{j-1}$ within $\mathfrak{U}_{j-1} \cap \mathfrak{X}_j$ and $\mathfrak{Z}$ is any supplementary space of $\mathfrak{X}_{j-1} \oplus \mathfrak{Y} (= \mathfrak{X}_j \cap \mathfrak{U}_{j-1})$ within $\mathfrak{X}_j$. Consider a basis $\boldsymbol{e}_{j,1}, \ldots, \boldsymbol{e}_{j,s}$ of $\mathfrak{Y}$ and $\boldsymbol{e}_{j,s+1}, \ldots, \boldsymbol{e}_{j,m_j}$ of $\mathfrak{Z}$. Let E_j be the associated matrix and write $E_j =: (E_{\mathfrak{Y}} | E_{\mathfrak{Z}})$ its decomposition according to this cut[7]. Since $M\mathfrak{X}_j \subset \mathfrak{X}_j + \mathfrak{U}_{j-1} = \mathfrak{Z} + \mathfrak{U}_{j-1}$, we have $M(\mathfrak{Y} + \mathfrak{Z}) \subset \mathfrak{Z} + \mathfrak{U}_{j-1}$. Thus, we may write

$$ME_j - E_{\mathfrak{Z}} R \in \mathfrak{U}_{j-1} \tag{26}$$

for some rectangular matrix R. Let us prove that R has maximal rank. Let $\mathfrak{V} \subset \mathfrak{Z}$ be the vector space spanned by the columns of $E_{\mathfrak{Z}} R$. Then, (26) implies that $M(\mathfrak{Y} + \mathfrak{Z}) \subset \mathfrak{V} + \mathfrak{U}_{j-1}$. Let $\boldsymbol{v} \in \mathfrak{Z}$. Then,

$$\begin{aligned} \boldsymbol{v} \in \mathfrak{Z} \subset \mathfrak{X}_j \subset M\mathfrak{X}_j + \mathfrak{U}_{j-1} \subset M\mathfrak{X}_{j-1} + M(\mathfrak{Y} + \mathfrak{Z}) + \mathfrak{U}_{j-1} \\ \subset \mathfrak{X}_{j-1} + \mathfrak{U}_{j-2} + \mathfrak{V} + \mathfrak{U}_{j-1} \subset \mathfrak{V} + \mathfrak{U}_{j-1} \end{aligned}$$

by Lemma 10. Thus, there exists $\boldsymbol{w} \in \mathfrak{V}$ such that $\boldsymbol{v} + \boldsymbol{w} \in \mathfrak{U}_{j-1}$. Meanwhile $\boldsymbol{v} + \boldsymbol{w} \in \mathfrak{Z}$, since $\mathfrak{V} \subset \mathfrak{Z}$. Since $\mathfrak{Z} \cap \mathfrak{U}_{j-1} = \{0\}$, $\boldsymbol{v} + \boldsymbol{w} = 0$, that is $\boldsymbol{v} = -\boldsymbol{w} \in \mathfrak{V}$. Hence $\mathfrak{V} = \mathfrak{Z}$, that is, R has maximal rank. Thus, one can complete R into an invertible matrix V. Then, since the columns of $E_{\mathfrak{Y}}$ belongs to $\mathfrak{U}_{j-1}$, we have

$$ME_j - E_j V \in \mathfrak{U}_{j-1}$$

We conclude by setting $\Theta_j := V^{-1}$. □

[7] If $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) is the zero space then $E_{\mathfrak{Y}}$ (resp. $E_{\mathfrak{Z}}$) is the empty matrix.

Algorithm 1: An algorithm to compute Θ and a truncation of P at order μ where (P, Θ) is an admissible pair.

Input: An integer $p \geq 2$ and a square matrix A with coefficients in $\mathbf{K}((z))$ such that $d(A) = 1$.

Output: A matrix Θ and a truncation of a matrix P up to order μ such that (P, Θ) is an admissible pair with respect to the system (6).

Set $\mathfrak{X} := \{0\}$. `// 𝔛 will successively be equal to` $\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \ldots$

Let $\mathcal{E}$ and $\mathcal{U}$ denote the empty lists. `// They will respectively host the matrices` $E_1, \ldots, E_r$ `and the spaces` $\mathfrak{U}_0, \ldots, \mathfrak{U}_{r-1}$.

while $\dim(\mathfrak{X}) < m$ **do**

  `// Inductive computation of` $\mathfrak{X}_1, \ldots, \mathfrak{X}_r$ `and of` $E_1, \ldots, E_r$.

  Set $\mathfrak{U} := \mathrm{span}_{\mathbf{K}}(M_k \mathfrak{X} \,:\, k \in \mathcal{S}'_p)$.

  Set $\mathfrak{F} := V_0$.

  Set $\mathfrak{G} := \mathfrak{F} \cap M^{-1}(\mathfrak{F} + \mathfrak{U}) \cap (M\mathfrak{F} + \mathfrak{U})$.

  while $\mathfrak{F} \neq \mathfrak{G}$ **do**

    Set $\mathfrak{F} := \mathfrak{G}$.

    Set $\mathfrak{G} := \mathfrak{F} \cap M^{-1}(\mathfrak{F} + \mathfrak{U}) \cap (M\mathfrak{F} + \mathfrak{U})$.

  end

  Let $\mathfrak{Y}$ be a supplementary space of $\mathfrak{X}$ within $\mathfrak{U} \cap \mathfrak{F}$.

  Let $\mathfrak{Z}$ be any supplementary space of $\mathfrak{X} \oplus \mathfrak{Y}$ within $\mathfrak{F}$.

  Let $E_{\mathfrak{Y}}$ (*resp.* $E_{\mathfrak{Z}}$) be a matrix whose columns form a basis of $\mathfrak{Y}$ (*resp.* $\mathfrak{Z}$) and set $E = (E_{\mathfrak{Y}} | E_{\mathfrak{Z}})$.

  Append E to the list $\mathcal{E}$ and $\mathfrak{U}$ to the list $\mathcal{U}$.

  Set $\mathfrak{X} := \mathfrak{F}$.

end

`// At this stage` $\mathcal{E} = (E_1, \ldots, E_r)$, $\mathcal{U} = (\mathfrak{U}_0, \ldots, \mathfrak{U}_{r-1})$ `and` $\mathfrak{X} = \mathfrak{X}_r$.

Set $r := \mathrm{card}(\mathcal{E})$.

for j from 1 to r **do**

  Compute an invertible matrix Θ_j with entries in $\mathbf{K}$ such that $E_j - ME_j\Theta_j \in \mathfrak{U}_{j-1}$.

  Let $\Theta_{i,j,k}$ be the constant matrices such that $E_j - ME_j\Theta_j = \sum_{i<j} \sum_{k \in \mathcal{S}'_p} M_k E_i \Theta_{i,j,k}$.

  Let $P_j(z) = \sum_{k=\nu_P}^{\mu} Q_{j,k} z^k$ be the matrix of Laurent polynomial of valuation at least ν_P and degree at most μ such that $\pi(P_j) = E_j$.

end

Let $\overline{P}(z) := (P_1(z) \mid \cdots \mid P_r(z))$ and let Θ be the block upper triangular matrix with diagonal blocks $\Theta_1, \ldots, \Theta_r$ and whose (i,j)th block is $\Theta_{i,j}(z) = \sum_{k \in \mathcal{S}'_p} \Theta_{i,j,k} z^k$ when $i < j$.

return $(\overline{P}, \Theta)$.

Algorithm 1 returns the truncation of P up to order μ. Indeed, it is not difficult to check that at the end of the first 'while' loop we have $\mathfrak{X} = \mathfrak{X}_r$ and the matrices $E_1, \ldots, E_r$ satisfy (20). Thus, the result follows from Proposition 7. Then, thanks to Remark 6, it is an easy task to compute coefficients of P with higher orders. Note that $r \leq m$, since the sequence $(\mathfrak{X}_i)_{1 \leq i \leq r}$ is increasing and, if we had $\mathfrak{X}_{i_0} = \mathfrak{X}_{i_0+1}$ for some $i_0 < r$, the sequence would be stationary from this index. Furthermore, for each j, Lemma 11 implies that $\ell_j \leq m(\mu - \nu_P + 1)$. Thus the second 'while' loop ends after

at most $m(\mu - \nu_P + 1)$ iterations. In the end, it is possible to bound the algorithm's complexity. We shall not provide details here.

5. Computing the Hahn and the constant part of a fundamental matrix of solutions

Throughout Section 5, we assume that $\mathbf{K}$ is an effective field which is algebraically closed. This restriction is necessary to obtain the matrix e_C in the required form. The matrix H could be computed over any field, but presenting it in this generality would significantly complicate the arguments; therefore, we restrict to the algebraically closed case. We fix an equation (23) for which $d(A) = 1$. We fix an admissible pair (P, Θ) for which we assume $P \in \mathrm{GL}_m(\mathbf{K}((z)))$ and $\Theta \in \mathrm{GL}_m(\mathbf{K}[z^{-1}])$. Up to a gauge transformation, we may suppose that Θ is upper triangular. Our goal is to describe the matrices H and e_C that intervene in the decomposition (7) of a fundamental matrix of solutions. Recall that they satisfy

$$\phi_p(H)C = \Theta H \quad \text{and} \quad \phi_p(e_C) = Ce_C$$

where C is the (upper triangular) constant term of Θ.

5.1. **Computing the matrix of Hahn series.** The general form (7) of a fundamental matrix of solutions guarantees that there exists a matrix $H \in \mathrm{GL}_m(\mathscr{H}_{\mathbf{K}})$ such that $\phi_p(H)C = \Theta H$, where C is the constant part of Θ. More precisely, it is established in [FR25] that one may take H to be upper triangular with only 1s on the diagonal and upper triangular entries with support in the set of negative rational numbers —a fact we will reprove here. Let $\theta_{i,j}(z) \in \mathbf{K}[z^{-1}]$, $i < j$, denote the upper diagonal entries of Θ and $\theta_1, \dots, \theta_m \in \mathbf{K}$ denote its diagonal entries. Let $c_{i,j}$, $i < j$, denote the upper diagonal entries of C, that is, the constant term of $\theta_{i,j}(z)$. An upper triangular matrix $H = (h_{i,j})_{i,j}$ with the identity on the diagonal and upper diagonal entries with support in $\mathbb{Q}_{<0}$ satisfies $\phi_p(H)C = \Theta H$ if and only if, for any i, j,

$$\theta_j h_{i,j}(z^p) - \theta_i h_{i,j}(z) = \sum_{k=i+1}^{j-1} \theta_{i,k}(z) h_{k,j}(z) + \theta_{i,j}(z) - c_{i,j} - \sum_{l=i+1}^{j-1} c_{l,j} h_{i,l}(z^p)\,. \tag{27}$$

The right-hand side of the equality has support in the set of negative rational numbers since $\theta_{i,j}(z) - c_{i,j} \in z^{-1}\mathbf{K}[z^{-1}]$. Consider the order $\prec$ on $\{(i,j) : i < j\}$ introduced in the proof of Proposition 5 and defined by $(k,l) \prec (i,j)$ if and only if either $l = j$ and $k > i$, or $l < j$. Then, the $h_{k,j}$ and $h_{i,l}$ intervening in the right-hand side are such that $(k,j) \prec (i,j)$ and $(i,l) \prec (i,j)$. Thus, the Hahn series $h_{i,j}$ may be computed by induction on the pairs (i,j).

By linearity, one only needs to consider inhomogeneous equations for which the second member has one term. We consider three types of equations. Precisely, given $\kappa, \eta, \tau \in \mathbf{K}^\times$, $\gamma \in \mathbb{Q}_{>0}$, $s \geq 1$ and $\boldsymbol{\omega} = (\boldsymbol{u}, \boldsymbol{a})$ with $\boldsymbol{u} = (u_{k_1,\dots,k_s})_{k_1,\dots,k_s} \subset \mathbf{K}$ a multi-linear recurrence sequence and $\boldsymbol{a} = (a_1, \dots, a_s)$ we consider the equations

$$\kappa h(z^p) - \eta h(z) = \tau z^{-\gamma} \tag{28}$$

$$\kappa h(z^p) - \eta h(z) = \tau z^{-\gamma} \boldsymbol{\xi}_{\boldsymbol{\omega}}(z) \tag{29}$$

$$\kappa h(z^p) - \eta h(z) = \tau \boldsymbol{\xi}_{\boldsymbol{\omega}}(z) \tag{30}$$

Recall that the Hahn series $\boldsymbol{\xi}_{\boldsymbol{\omega}}$ are defined by

$$\boldsymbol{\xi}_{\boldsymbol{\omega}}(z) := \sum_{k_1,\cdots,k_s \geq 1} u_{k_1,\dots,k_s} z^{-\frac{a_1}{p^{k_1}} - \cdots - \frac{a_s}{p^{k_1+\cdots+k_s}}}, \quad \text{with } \boldsymbol{\omega} := (\boldsymbol{u}, \boldsymbol{a}),$$

where $\boldsymbol{u} := (u_{k_1,\dots,k_s})_{k_1,\dots,k_s} \subset \mathbf{K}$ is a multi-linear recurrence sequence and $\boldsymbol{a} := (a_1,\dots,a_s) \in \mathbb{Q}_{>0}$ for some integer $s \geq 0$.

Let us recall some facts about multi-linear recurrence sequences. We follow the approach from [Sch00]. Let $s \geq 1$ be an integer and $\boldsymbol{X} = (X_1,\dots,X_s)$ be a s-tuple of indeterminates. The ring $\mathbf{K}[\boldsymbol{X}^{\pm 1}] = \mathbf{K}[X_1^{\pm 1},\dots,X_s^{\pm 1}]$ acts on the $\mathbf{K}$-vector space of sequences $\mathbf{K}^{\mathbb{Z}^s}$ by

$$(P\boldsymbol{u})_{k_1,\dots,k_s} := \sum_{i_1,\dots,i_s \in \mathbb{Z}} p_{i_1,\dots,i_s} u_{k_1+i_1,\dots,k_s+i_s}$$

when $P(\boldsymbol{X}) := \sum_{i_1,\dots,i_s} p_{i_1,\dots,i_s} X_1^{i_1} \cdots X_s^{i_s}$ and $\boldsymbol{u} := (u_{k_1,\dots,k_s})_{k_1,\dots,k_s}$. A sequence $\boldsymbol{u} \in \mathbf{K}^{\mathbb{Z}^s}$ is multi-linear recurrent if, letting $\mathcal{I}(\boldsymbol{u})$ be the ideal $\{P \in \mathbf{K}[\boldsymbol{X}^{\pm 1}] \,:\, P\boldsymbol{u} = 0\}$, the quotient $\mathbf{K}[\boldsymbol{X}^{\pm 1}]/\mathcal{I}(\boldsymbol{u})$ is a finite dimensional $\mathbf{K}$-vector space. It is actually equivalent requiring that $\boldsymbol{u}$ is linear recurrent with respect to each of its variables.

Lemma 13. *A sequence $\boldsymbol{u} = (u_{k_1,\dots,k_s})_{k_1,\dots,k_s} \in \mathbf{K}^{\mathbb{Z}^s}$ is multi-linear recurrent if and only if, for each $i \in \{1,\dots,s\}$ there exist an integer $m_i \geq 1$ and coefficients $b_{i,0},\dots,b_{i,m_i}$, with $b_{i,m_i} b_{i,0} \neq 0$ such that, for every integers $k_1,\dots,k_s \in \mathbb{Z}$,*

$$\sum_{j=0}^{m_i} b_{i,j} u_{k_1,\dots,k_{i-1},k_i+j,k_{i+1},\dots,k_s} = 0\,. \tag{31}$$

Proof. We first prove the reverse direction. Identity (31) is equivalent to the fact that the polynomials $\sum_{j=0}^{m_i} b_{i,j} X_i^j$ belong to $\mathcal{I}(\boldsymbol{u})$. Thus, the monomials $X_1^{j_1} \cdots X_s^{j_s}$, $j_i \in \{0,\dots,m_i-1\}$, $i \in \{1,\dots,s\}$, span the vector space $\mathbf{K}[\boldsymbol{X}^{\pm 1}]/\mathcal{I}(\boldsymbol{u})$. *A fortiori*, it is finite dimensional. Thus $\boldsymbol{u}$ is multi-linear recurrent.

Conversely, suppose that $\boldsymbol{u}$ is multi-linear recurrent and let m be the dimension of $\mathbf{K}[\boldsymbol{X}^{\pm 1}]/\mathcal{I}(\boldsymbol{u})$. Then, for each i, $1, X_i, X_i^2, \dots, X_i^m \in \mathbf{K}[\boldsymbol{X}^{\pm 1}]$ are linearly dependent modulo $\mathcal{I}(\boldsymbol{u})$. Thus, (31) holds for some $m_i \leq m$. □

Given a multi-linear recurrence sequence $\boldsymbol{u}$ and a element $\theta \in \mathbf{K}$, the sequence with general term

$$u^{[\theta]}_{l,k_2,\dots,k_s} := \sum_{k_1=1}^{l-1} u_{k_1,\dots,k_s} \theta^{l-k_1}$$

is a multi-linear recurrence sequence. We denote it by $\boldsymbol{u}^{[\theta]}$. Indeed, let (31) be the equations satisfied by $\boldsymbol{u}$. The sequence $\boldsymbol{u}^{[\theta]}$ also satisfies these equations when $i \in \{2,\dots,s\}$. When $i = 1$, since

$$u^{[\theta]}_{l+1,k_2,\dots,k_s} - \theta u^{[\theta]}_{l,k_2,\dots,k_s} = u_{l,\dots,k_s}$$

we have

$$b_{1,m} u^{[\theta]}_{l+m+1,k_2,\dots,k_s} + \sum_{j=1}^{m} (b_{1,j-1} - \theta b_{1,j}) u^{[\theta]}_{l+1,k_2,\dots,k_s} - \theta u^{[\theta]}_{l,k_2,\dots,k_s} = 0.$$

Using Lemma 13 we conclude that $\boldsymbol{u}^{[\theta]}$ is a multi-linear recurrence sequence.

We are now ready to solve Equations (28), (29) and (30).

Lemma 14. *Let $\kappa, \eta, \tau \in \mathbf{K}^\times$ and $\gamma \in \mathbb{Q}_{>0}$. Let $s \geq 1$ and $\boldsymbol{\omega} := (\boldsymbol{u}, \boldsymbol{a})$ with $\boldsymbol{u} := (u_{k_1,\dots,k_s})_{k_1,\dots,k_s}$ and $\boldsymbol{a} := (a_1,\dots,a_s)$. Then:*

(i) *$h(z) := \frac{\tau}{\eta} \boldsymbol{\xi}_{((u_k)_k,(\gamma))}(z)$ is a solution to (28), where $u_k := (\eta\kappa^{-1})^k$;*

(ii) $h(z) := \frac{\tau}{\eta}\boldsymbol{\xi}_{(\boldsymbol{v},(\gamma,a_1,\dots,a_s))}(z)$ *is a solution to* (29) *where* $\boldsymbol{v} := (v_{k_0,\dots,k_s})_{k_0,\dots,k_s}$ *with*

$$v_{k_0,\dots,k_s} := (\eta\kappa^{-1})^{k_0} u_{k_1,\dots,k_s}$$

(iii) $h(z) := \frac{\tau}{\eta}\boldsymbol{\xi}_{(\boldsymbol{u}^{[\eta\kappa^{-1}]},\boldsymbol{a})}(z)$ *is a solution to* (30).

Proof. The proof of (i) is straightforward. Proving that the series from (ii) satisfies (29) is straighforward too. The fact that the sequence $\boldsymbol{v}$ is multi-linear recurrent is an immediate consequence of Lemma 13. Let us prove (iii). It is not difficult to check that the series

$$h = \frac{\tau}{\eta}\sum_{k\geq 1}\left(\frac{\eta}{\kappa}\right)^k \phi_p^{-k}(\boldsymbol{\xi}_{\boldsymbol{\omega}})$$

satisfies

$$\kappa h(z^p) - \eta h(z) = \tau\boldsymbol{\xi}_{\boldsymbol{\omega}}(z).$$

The fact that $h(z)$ is a well-defined Hahn series is a consequence of [Roq24, Lemma 33]. It only remains to prove that $h(z)$ has the desired form. Making a change of variable $l \leftarrow k + k_1$ and setting $\theta := \eta\kappa^{-1}$, we have

$$\begin{aligned}
h(z) &= \frac{\tau}{\eta}\sum_{k\geq 1}\theta^k\phi_p^{-k}(\boldsymbol{\xi}_{\boldsymbol{\omega}}) \\
&= \frac{\tau}{\eta}\sum_{k,k_1,\dots,k_s\geq 1} u_{k_1,\dots,k_s}\theta^k z^{-\frac{a_1}{p^{k+k_1}}-\dots-\frac{a_s}{p^{k+k_1+\dots+k_s}}} \\
&= \frac{\tau}{\eta}\sum_{l\geq 2}\left(\sum_{k_1=1}^{l-1} u_{k_1,\dots,k_s}\theta^{l-k_1}\right) z^{-\frac{a_1}{p^l}-\frac{a_2}{p^{l+k_2}}-\dots-\frac{a_s}{p^{l+k_2+\dots+k_s}}} \\
&= \frac{\tau}{\eta}\sum_{l,k_2,\dots,k_s\geq 1} u^{[\theta]}_{l,k_2,\dots,k_s} z^{-\frac{a_1}{p^l}-\frac{a_2}{p^{l+k_2}}-\dots-\frac{a_s}{p^{l+k_2+\dots+k_s}}} \\
&= \frac{\tau}{\eta}\boldsymbol{\xi}_{(\boldsymbol{u}^{[\theta]},\boldsymbol{a})}(z)\,.
\end{aligned}$$

□

This enables us to compute the matrix H explicitly.

Algorithm 2: An algorithm to compute the matrix H

Input: An upper triangular matrix Θ, with non-zero constant diagonal entries and upper diagonal entries in $\mathbf{K}[z^{-1}]$

Output: A matrix H with coefficients in $\mathrm{span}_{\mathbf{K}}\{\boldsymbol{\xi}_{\boldsymbol{\omega}} : \boldsymbol{\omega}\in\Omega\}$ such that $\phi_p(H)C = \Theta H$

Set $h_{i,j} := 0$ for $1 \leq j < i \leq m$ and $h_{i,i} := 1$ for every $i \in \{1,\dots,m\}$;

for $(i,j) \in \{1 \leq i < j \leq m\}$ *ordered with respect to* $\prec$ **do**

  Using the expressions of $h_{k,l}$, $(k,l) \prec (i,j)$ previously computed, decompose (27) as a finite sum of equations of the form (28), (29) and (30).

  Solve each of these equations by the formulas given in Lemma 14.

  Set $h_{i,j}$ to be the sum of these solutions.

end

return $H := (h_{i,j})_{i,j}$.

Remark. When $\mathbf{K}$ has characteristic 0, the sequences with general term

$$k_1^{\alpha_1}\cdots k_s^{\alpha_s}\theta_1^{k_1}\cdots\theta_s^{k_s},\quad \alpha_1,\dots,\alpha_s\in\mathbb{Z}_{\geq 0},\ \theta_1,\dots,\theta_s\in\mathbf{K}^\times \tag{32}$$

form a basis of the $\mathbf{K}$-vector space of multi-linear recurrence sequences [Sch00, Theorem 2.1, Lemma 2.2]. In that case, the set $\{\boldsymbol{\xi}_{\boldsymbol{\omega}},\ \boldsymbol{\omega}\in\Omega\}$ coincides with the $\mathbf{K}$-vector space spanned by the Hahn series introduced in [FR25]. It is not difficult to adapt Algorithm 2 so that it returns the entries of H as linear combinations of some $\boldsymbol{\xi}_{(\boldsymbol{u},\boldsymbol{a})}$ with $\boldsymbol{u}$ of the form (32). Indeed, using Lemma 14, the only delicate task is to express $\boldsymbol{u}^{[\theta]}$ under this form when the general term of $\boldsymbol{u}$ satisfies (32). Under this assumption, the general term of $\boldsymbol{u}^{[\theta]}$ is

$$u^{[\theta]}_{l,k_2,\dots,k_s}=\left(\sum_{k_1=1}^{l-1}k_1^{\alpha_1}(\theta_1\theta^{-1})^{k_1}\right)k_2^{\alpha_2}\cdots k_s^{\alpha_s}\theta^l\theta_2^{k_2}\cdots\theta_s^{k_s}\,.$$

Thus, one only has to express $\sum_{k=1}^{l-1}k^{\alpha_1}(\theta_1\theta^{-1})^k$ under the desired form. When $\theta_1\theta^{-1}=1$, Faulhaber's formula describes it as an explicit polynomial of degree α_1+1 in l. When $\theta_1\theta^{-1}\neq 1$, one can find a polynomial $P(X)$ and a number $\gamma\in\mathbf{K}$, depending on α_1,θ_1,θ, such that $\sum_{k=1}^{l-1}k^{\alpha_1}(\theta_1\theta^{-1})^k=P(l)(\theta_1\theta^{-1})^l+\gamma$. Explicit formulas may easily be found by applying α_1 times the operator $\lambda\frac{\partial}{\partial\lambda}$ on both sides of the identity $\sum_{k=1}^{l-1}\lambda^k=\frac{\lambda^l-1}{\lambda-1}$ and setting $\lambda:=\theta_1\theta^{-1}$.

To conclude this section, and for the sake of completeness, let us explain why the Hahn series $\boldsymbol{\xi}_{\boldsymbol{\omega}}$ are solutions of p-Mahler equations. Since this result is not necessary to prove Theorem 1 and Theorem 2, we shall be brief. When the characteristic of $\mathbf{K}$ is 0, this result is proved in [FR25] using the fact that any multi-linear recurrence sequence is a linear combination of sequences with general term $k_1^{\alpha_1}\cdots k_s^{\alpha_s}\lambda_1^{k_1}\cdots\lambda_s^{k_s}$. This approach does not work in positive characteristic (consider, for example, the sequence $\lfloor k/p\rfloor \mod p$ in $\mathbb{F}_p$). Fix a Hahn series $\boldsymbol{\xi}_{\boldsymbol{\omega}}$. Let s be the associated parameter. If $s=0$ then $\boldsymbol{\xi}_{\boldsymbol{\omega}}=1$ is a solution of a p-Mahler equation. Suppose that $s\geq 1$ and that the result holds for any $\boldsymbol{\xi}_{\boldsymbol{\omega}'}$ with $s'<s$. Keeping the notation of Lemma 13, it is straightforward that

$$b_{1,0}\boldsymbol{\xi}_{\boldsymbol{\omega}}(z)+b_{1,1}\boldsymbol{\xi}_{\boldsymbol{\omega}}(z^p)+\cdots+b_{1,m_1}\boldsymbol{\xi}_{\boldsymbol{\omega}}\left(z^{p^{m_1}}\right)$$

is a linear combination of some $z^{-\gamma}\boldsymbol{\xi}_{\boldsymbol{\omega}'}$ where the parameter s' associated to each $\boldsymbol{\omega}'$ is $s-1$. By induction hypothesis, $\boldsymbol{\xi}_{\boldsymbol{\omega}}$ is a solution of an inhomogeneous equation whose right-hand side is itself a solution of a p-Mahler equation. It is classical that $\boldsymbol{\xi}_{\boldsymbol{\omega}}$ is then a solution of a p-Mahler equation.

5.2. **Computing solutions of constant systems.** Let C denote a non-singular constant matrix. The last step consists in computing a matrix e_C whose entries are $\mathbf{K}$-linear combinations of the $e_c\ell^j$ such that $\phi_p(e_C)=Ce_C$. The strategy is explicitly described in [FR25]. Let us recall it briefly. Let $C=DU$ be the multiplicative Dunford-Jordan decomposition of C, where D is diagonalizable, U is unipotent, and D and U commute. Set $\ell^{[k]}:=\frac{\ell(\ell-1)\cdots(\ell-k+1)}{k!}$ when $k\in\mathbb{Z}_{\geq 0}$. Then, since $(U-\mathrm{I})^m=0$, the matrix

$$e_U:=\sum_{k=0}^{m-1}\ell^{[k]}(U-\mathrm{I})^k$$

satisfies $\phi_p(e_U)=Ue_U$. Let $Q\in\mathrm{GL}_m(\mathbf{K})$ be such that $QDQ^{-1}=\mathrm{diag}(c_1,\dots,c_m)$, with $c_1,\dots,c_m\in\mathbf{K}^\times$. Then, $e_D:=Q\mathrm{diag}(e_{c_1},\dots,e_{c_m})Q^{-1}$ satisfies $\phi_p(e_D)=De_D$.

Eventually, since D and U commute, e_U and e_D commute and $e_C := e_D e_U$ satisfies $\phi_p(e_C) = Ce_C$. This leads to the following algorithm.

Algorithm 3: An algorithm to compute a fundamental matrix of solution to a constant system

Input: A constant matrix C with coefficients in $\mathbf{K}$.
Output: The matrix e_C.
Write $C = DU$;
Set $e_U := \sum_{k=0}^{m-1} \ell^{[k]}(U - \mathrm{I})^k$;
Let $Q \in \mathrm{GL}_m(\mathbf{K})$ be such that $QDQ^{-1} = \mathrm{diag}(c_1, \ldots, c_m)$ is diagonal;
Set $e_D := Q\mathrm{diag}(e_{c_1}, \ldots, e_{c_m})Q^{-1}$;
return $e_D e_U$;

5.3. **On the necessity for the base field to be algebraically closed.** We claim that, when $\mathbf{K}$ is not algebraically closed, it is not always possible to build a matrix e_C whose entries are linear combinations of some $e_c\ell$ with $c \in \mathbf{K}^\times$. Let us justify this. Over the base field $\mathbb{Q}$ we consider the matrix

$$C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Algorithm 3 returns the matrix

$$e_C := \begin{pmatrix} \frac{1}{2}(e_i + e_{-i}) & \frac{i}{2}(e_{-i} - e_i) \\ \frac{i}{2}(e_i - e_{-i}) & \frac{1}{2}(e_i + e_{-i}) \end{pmatrix}$$

Since the elements e_c, $c \in \overline{\mathbb{Q}}$, are linearly independent over $\mathscr{H}_{\overline{\mathbb{Q}}}[\ell]$ and since the solution e_C of the system $\phi_p(Y) = CY$ is unique up to right product by an element of $\mathrm{GL}_m(\overline{\mathbb{Q}})$, there exists no such solution whose entries are linear combinations of the elements $e_c\ell^j$, $c \in \mathbb{Q}$, $j \in \mathbb{Z}_{\geq 0}$.

5.4. **Description of the main algorithm.** Recall that, to any equation (2), we can associate a Mahler system by considering the companion matrix

$$A := \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\frac{a_0}{a_m} & \cdots & \cdots & -\frac{a_{m-1}}{a_m} \end{pmatrix}.$$

Then, finding a basis of solutions to (2) amounts to computing a fundamental matrix of solutions to the associated system. This is the strategy which justifies Algorithm 4.

Algorithm 4: An algorithm to compute a basis of solution to a Mahler equation

Input: A Mahler equation (2) and an integer n
Output: A basis of solution in the form (5), with a truncation of the series $f_{i,c,j,\boldsymbol{\omega}}$ up to order n

```
Let A denote the companion matrix associated to this equation;
Compute d := d(A);
if d ≠ 1 then
    replace A with A(z^d);
end
Let (P̄, Θ) be the output of Algorithm 1 when taking A as input;
Compute the coefficients P_k of the Puiseux matrix P up to order dn thanks to (17);
Replace P̄ with this new matrix;
Compute Q ∈ GL_m(K̄) such that QΘQ^{-1} is upper triangular;
Replace P̄ with P̄Q^{-1} and Θ with QΘQ^{-1};
Let H be the output of Algorithm 2 taking Θ as input;
Let e_C be the output of Algorithm 3 taking the constant term of Θ as input;
return the first row of P̄(z^{1/d})H(z^{1/d})e_C.
```

Remark. In the last step of this algorithm, one has to deal with terms of the form $\boldsymbol{\xi}_{\boldsymbol{\omega}}(z^{1/d})$. However, up to replace the tuple $\boldsymbol{a}$ with $\boldsymbol{a}/d$, such a term is of the form $\boldsymbol{\xi}_{\boldsymbol{\omega}'}$ for some $\boldsymbol{\omega}' \in \Omega$. Thus, Algorithm 4 returns a basis of solutions of the desired form.

As already mentioned, the decomposition (5) is not unique. Nevertheless, when $\mathbf{K}$ has characteristic 0, it is possible to make it unique. Consider the set Ω_0 of pairs $\boldsymbol{\omega} := (\boldsymbol{u}, \boldsymbol{a})$ where

- $\boldsymbol{u}$ has a general term of the form $k_1^{\alpha_1} \cdots k_s^{\alpha_s} \lambda_1^{k_1} \cdots \lambda_s^{k_s}$, where $\alpha_1, \ldots, \alpha_s \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \ldots, \lambda_s \in \overline{\mathbf{K}}^{\times}$;
- a is a tuple of positive rational numbers whose denominator is coprime with p and whose numerator is not divisible by p (recall that p is not necessarily prime).

A decomposition (5) for which each $\boldsymbol{\omega}$ belongs to Ω_0 is called *standard* in [FR25] and such a decomposition always exists. Passing from any decomposition to the standard one is not difficult. Details are provided in [FR25, Section 5.1.2].

5.5. **Computing an equation for each Puiseux series appearing in (5).** Algorithm 4 returns a list $(y_1, \ldots, y_m)$ of solutions of a p-Mahler equation (11), of the form

$$y_i = \sum_{c \in K_0} \sum_{j=0}^{j_0} \sum_{\boldsymbol{\omega} \in \Omega_1} f_{i,c,j,\boldsymbol{\omega}}(z) \boldsymbol{\xi}_{\boldsymbol{\omega}} e_c \ell^j$$

where K_0, Ω_1 are finite sets and $f_{i,c,j,\boldsymbol{\omega}}$ are Puiseux series whose expansion up to z^n is computed. As mentioned in the introduction, each series can be uniquely determined by providing a p-Mahler equation it satisfies. The construction is as follows. These Puiseux series $f_{i,c,j,\boldsymbol{\omega}}$ are explicit $\overline{\mathbf{K}}$-linear combinations of the entries of the matrix P, the first matrix of an admissible pair (P, Θ). The equation satisfied by a sum of p-Mahler Puiseux series can be easily determined from the equations satisfied by each term. Hence, one only has to compute an equation for each entry of P. Let $f_{i,j}$ be the

(i,j)-th entry of P. From $\phi_p(P)\Theta = AP$ we deduce that for each k,

$$f_{i,j}(z^{p^k}) = \left(\phi_p^{k-1}(A)\cdots\phi_p(A)AP\Theta^{-1}\phi_p(\Theta)^{-1}\cdots\phi_p^{k-1}(\Theta)^{-1}\right)_{i,j}$$

Thus, the m^2+1 series $f_{i,j}(z),\dots,f_{i,j}\left(z^{p^{m^2}}\right)$ are explicit linear combinations of the m^2 entries of P. One readily deduces a p-Mahler equation satisfied by $f_{i,j}$.

6. Examples

Let us first run Algorithm 4 on an example in characteristic 0. Consider the Rudin-Shapiro equation

$$y(z) + (z-1)y(z^2) - 2zy(z^4) = 0\,. \tag{33}$$

Here we have $p=2$ and we may take $\mathbf{K} = \overline{\mathbb{Q}}$. We run Algorithm 4 with this equation as input and the integer $n := 9$. It computes the companion matrix

$$A(z) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2z} & \frac{z-1}{2z} \end{pmatrix}.$$

associated with (33). The associated Newton polygon is the convex hull of the set $\{(1,i),(2,j),(4,k) \,:\, i,j\geq 0,\ k\geq 1\}$. Its slopes are 0 and $\frac{1}{2}$ so that $d(A)=1$.

Our algorithm calls Algorithm 1 which returns a pair $(\overline{P},\Theta)$ with

$$\overline{P} = \begin{pmatrix} 1+z & z \\ 1 & \frac{1}{z}-1+z \end{pmatrix}, \quad \text{and} \quad \Theta = \begin{pmatrix} 1 & \frac{1}{z}-1 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

as we will prove in Section 6.1. Letting $\overline{P_1},\overline{P_2}$ denote the columns of $\overline{P}$, our algorithm then computes recursively the coefficients $P_{1,n}$ and $P_{2,n}$ for $n\in\{2,\dots,9\}$ thanks to (17). Then it replaces the matrix $\overline{P}$ with the matrix

$$\begin{pmatrix} 1+z+z^2-z^3+z^4+z^5-z^6+z^7+z^8+z^9 & z-\frac{5}{2}z^2+\frac{3}{2}z^3+\frac{5}{4}z^4-\frac{7}{4}z^5+\frac{5}{4}z^6-\frac{1}{4}z^7-\frac{5}{8}z^8+\frac{3}{8}z^9 \\ 1+z^2+z^4-z^6+z^7+z^9 & \frac{1}{z}-1+z-\frac{3}{2}z^2+z^3+\frac{1}{4}z^4-z^5+\frac{1}{4}z^6+z^7-\frac{13}{8}z^8+z^9 \end{pmatrix}$$

It then calls Algorithms 2 and 3, which return (see Sections 6.2 and 6.3)

$$H = \begin{pmatrix} 1 & \boldsymbol{\xi}_{([-2],1)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad e_C = \begin{pmatrix} 1 & \frac{2}{3}e_{\frac{-1}{2}} - \frac{2}{3} \\ 0 & e_{-\frac{1}{2}} \end{pmatrix},$$

where $[-2] = ((-2)^k)_k \in \overline{\mathbb{Q}}^{\mathbb{Z}}$. Then, Algorithm 4 returns the entries of the first row of $\overline{P}He_C$, that is

$$f_1 := 1+z+z^2-z^3+z^4+z^5-z^6+z^7+z^8+z^9$$

and $f_2 := -\frac{2}{3}f_1 + \left(f_1\boldsymbol{\xi}_{([-2],1)} + \frac{2}{3}f_1 + g\right)e_{-\frac{1}{2}}$ where g is the top-right entry of $\overline{P}$.

6.1. Calling Algorithm 1.

Since $\operatorname{val} A = -1$ and $\operatorname{val} A^{-1} = 0$, the parameters (15) are $\nu_P = -1$, $\nu_\Theta = -1$, $\nu = -3$, and $\mu = 1$. Write $A^{-1}(z) = B_0 + B_1 z$ with

$$B_0 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_1 := \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}.$$

One checks that $M = M_0$ and M_1 have the following block-decomposition

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & B_0 & 0 \\ 0 & 0 & 0 & B_1 & 0 \end{pmatrix} \quad \text{and} \quad M_{-1} = \begin{pmatrix} 0 & 0 & B_0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & B_0 & 0 \\ 0 & 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 & B_0 \end{pmatrix}$$

where 0 denotes the zero matrix of size 2×2. Algorithm 1 is initialized with $\mathfrak{X} := \{0\}$, $\mathcal{E} := ()$ and $\mathcal{U} := ()$.

First iteration of the main 'while' loop. Entering the main 'while' loop, it sets $\mathfrak{U} := \{0\}$, $\mathfrak{F} := V_0$ and computes

$$\mathfrak{G} := V_0 \cap M^{-1}V_0 \cap MV_0 = \{{}^{\mathrm{t}}(0,0,0,0,0,0,\lambda_1,\lambda_1,\lambda_2,0) \,:\, \lambda_1,\lambda_2 \in \mathbb{Q}\}$$

►*Execution of the inner 'while' loop*

- Since $\mathfrak{G} \neq \mathfrak{F}$, it enters the inner 'while' loop and it sets $\mathfrak{F} := \mathfrak{G}$. Then, it computes the new space $\mathfrak{G}$:

$$\mathfrak{G} := \mathfrak{F} \cap M^{-1}\mathfrak{F} \cap M\mathfrak{F} = \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_1\}.$$

  where $\boldsymbol{e}_1 := {}^{\mathrm{t}}(0,0,0,0,0,0,1,1,1,0)$.
- Since $\mathfrak{G} \neq \mathfrak{F}$, it enters this inner loop a second time. The computation at that stage leaves $\mathfrak{G}$ unchanged. Thus, our algorithm exits this loop.

Since $\mathfrak{X} = \mathfrak{U} = \{0\}$, it sets $\mathfrak{Y} := \{0\}$ and $\mathfrak{Z} := \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_1\}$. Thus $E_{\mathfrak{Y}}$ is the empty matrix and $E = E_{\mathfrak{Z}} = \boldsymbol{e}_1$. The algorithm appends $\boldsymbol{e}_1$ to $\mathcal{E}$ and the zero space to $\mathcal{U}$ so that $\mathcal{E} = (\boldsymbol{e}_1)$ and $\mathcal{U} = (\{0\})$. Then it sets $\mathfrak{X} = \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_1\}$. Since $\dim(\mathfrak{X}) = 1 < 2$, it enters a second time the main 'while' loop.

Second iteration of the main 'while' loop. It sets $\mathfrak{U} := \mathrm{span}_{\mathbb{Q}}(M_0\boldsymbol{e}_1, M_1\boldsymbol{e}_1)$ which is the $\mathbb{Q}$-vector space spanned by $\boldsymbol{e}_1$ and $\boldsymbol{x}$ where

$$\boldsymbol{x} := {}^{\mathrm{t}}(0,0,0,0,1,1,1,0,1,1)\,.$$

Then it sets $\mathfrak{F} := V_0$. Then, the first computation of $\mathfrak{G}$ gives

$$\mathfrak{G} := \{{}^{\mathrm{t}}(0,0,0,0,\lambda_0,\lambda_1,\lambda_1+\lambda_2,\lambda_2,\lambda_3,\lambda_1) \,:\, \lambda_0,\lambda_1,\lambda_2,\lambda_3 \in \mathbb{Q}\}$$

►*Execution of the inner 'while' loop*

- Since $\mathfrak{G} \neq \mathfrak{F}$, it enters the inner 'while' loop and it sets $\mathfrak{F} := \mathfrak{G}$. Then, $\mathfrak{G}$ becomes

$$\mathfrak{G} := \{{}^{\mathrm{t}}(0,0,0,0,0,\lambda_1,\lambda_1+\lambda_2,\lambda_2,\lambda_3,\lambda_1) \,:\, \lambda_1,\lambda_2,\lambda_3 \in \mathbb{Q}\}\,.$$

- One more iteration of this 'while' loop gives $\mathfrak{G} := \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ with

$$\boldsymbol{e}_2 := {}^{\mathrm{t}}(0,0,0,0,0,1,0,-1,1,1).$$

- Then, one last iteration leaves $\mathfrak{G}$ unchanged, so that it exits this loop.

Since $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{x}$ are linearly independent, we have $\mathfrak{Y} = \{(0)\}$ and $\mathfrak{Z} = \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_2\}$. Thus $E_{\mathfrak{Y}}$ is the empty matrix and $E = E_{\mathfrak{Z}} = \boldsymbol{e}_2$. Then, the algorithm appends $\boldsymbol{e}_2$ to $\mathcal{E}$ and the space $\mathfrak{U}$ to $\mathcal{U}$ so that $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $\mathcal{U} = (\{0\}, \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_1, \boldsymbol{x}\})$. Then it sets $\mathfrak{X} := \mathrm{span}_{\mathbb{Q}}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Since $\dim X = 2$, it leaves the main 'while' loop.

Execution of the 'for' loop. We have $r = 2$. The algorithm enters the 'for' loop. All blocks of Θ will be 1×1 matrices. By some abuse of notation we will write them as elements of $\mathbb{Q}$. When $j = 1$, the algorithm sets $\Theta_1 := 1$ and

$$\overline{P_1} = \begin{pmatrix} 1+z \\ 1 \end{pmatrix}.$$

In the second iteration of the loop (with $j = 2$), it sets $\Theta_2 := -1/2$, $\Theta_{1,2,0} := -1$, $\Theta_{1,2,-1} := 1$ and

$$\overline{P_2} = \begin{pmatrix} z \\ \frac{1}{z} - 1 + z \end{pmatrix}.$$

It exits the 'for' loop and returns $((\overline{P_1}, \overline{P_2}), \Theta)$ where

$$\Theta := \begin{pmatrix} 1 & \frac{1}{z} - 1 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

6.2. **Calling Algorithm 2.** Algorithm 2 takes as input the matrix Θ. The only entry of H this algorithm has to compute is $h_{1,2}$ which satisfies

$$-\frac{1}{2} h_{1,2}(z^2) - h_{1,2}(z) = \frac{1}{z}.$$

This is an equation of the form (28) with $\kappa := -\frac{1}{2}$ and $\eta := \tau := 1$. Thanks to Lemma 14, it returns $h_{1,2}(z) := \boldsymbol{\xi}_{([-2],1)} = \sum_{k \geq 1} (-2)^k z^{-1/2^k}$ and

$$H := \begin{pmatrix} 1 & \boldsymbol{\xi}_{([-2],1)} \\ 0 & 1 \end{pmatrix}.$$

6.3. **Calling Algorithm 3.** Algorithm 3 is called with $C =: \begin{pmatrix} 1 & -1 \\ 0 & -\frac{1}{2} \end{pmatrix}$ as an input. The DU decomposition of C is $U := \mathrm{Id}$ and $D := C$. After diagonalizing C, Algorithm 3 returns e_C where

$$e_C = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e_{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{3} e_{\frac{-1}{2}} - \frac{2}{3} \\ 0 & e_{-\frac{1}{2}} \end{pmatrix}.$$

6.4. **An equation for the second Puiseux term.** Algorithm 4 returns f_1 and f_2, which are truncations at order $n = 9$ of two solutions h_1 and h_2 to (33). The solution h_1 is a power series solution to (33). By contrast, the solution h_2 is of the form

$$h_2 = -\frac{2}{3} h_1 + \left(h_1 \boldsymbol{\xi}_{([-2],1)} + \frac{2}{3} h_1 + h \right) e_{-\frac{1}{2}}$$

where h is some power series. Precisely, if P is the matrix of Laurent series whose truncation is $\overline{P}$ and who satisfies $\phi_p(P)\Theta = AP$, then, h is the top-right entry of P. Let us explain how to compute a Mahler equation satisfied by h. We let $P_{i,j}$ denote the entries of P. In particular, $h_1 = P_{1,1}$ and $h = P_{1,2}$. Iterating the identity $\phi_p(P)\Theta = AP$ and isolating the terms $h(z^{2^k})$, $k \in \{0, \ldots, 4\}$, we obtain

$$\begin{pmatrix} h(z) \\ h(z^2) \\ h(z^4) \\ h(z^8) \\ h(z^{16}) \end{pmatrix} = \left(\begin{array}{c|c|c|c}
0 & 1 & 0 & 0 \\ \hline
0 & 0 & \frac{2(1-z)}{z} & -2 \\ \hline
\frac{(z-1)^2}{z^3} & \frac{2}{z} & \frac{(z-1)^3}{z^3} & \frac{2(z-1)}{z} \\ \hline
\begin{array}{c} \frac{-3z^6+4z^5+z^4}{2z^7} \\ \frac{-4z^3+3z^2-1}{2z^7} \end{array} & \frac{2(1-z^2)}{z^3} & \begin{array}{c} \frac{-3z^7+7z^6-9z^5+3z^4}{2z^7} \\ \frac{+3z^3-3z^2+z+1}{2z^7} \end{array} & \frac{-2(z^3-z^2+z+1)}{z^3} \\ \hline
\begin{array}{c} \frac{5z^{14}-8z^{13}-z^{12}+8z^{11}}{4z^{15}} \\ \frac{-z^{10}-8z^9+11z^8-8z^7}{4z^{15}} \\ \frac{+3z^6-3z^4+z^2+1}{4z^{15}} \end{array} & \frac{2(z^6-z^4+z^2+1)}{z^7} & \begin{array}{c} \frac{5z^{15}-13z^{14}+17z^{13}-7z^{12}}{4z^{15}} \\ \frac{-z^{11}-7z^{10}+5z^9-3z^8}{4z^{15}} \\ \frac{+3z^7-3z^6+3z^5+3z^4}{4z^{15}} \\ \frac{+z^3-z^2-z-1}{4z^{15}} \end{array} & \begin{array}{c} \frac{2z^7-2z^6+2z^5}{z^7} \\ \frac{+2z^4+2z^3}{z^7} \\ \frac{-2z^2-2z-2}{z^7} \end{array}
\end{array}\right) \begin{pmatrix} P_{1,1}(z) \\ P_{1,2}(z) \\ P_{2,1}(z) \\ P_{2,2}(z) \end{pmatrix}$$

Looking at a vector in the left-kernel of this matrix, we get

$$\begin{aligned}(-2z^{10}-6z^9-8z^8-8z^7-4z^6+4z^5+8z^4+8z^3+6z^2+2z)h(z)\\ +(-z^{11}-z^9-2z^8+2z^7+2z^5+4z^4-z^3-z-2)h(z^2)\\ +(z^{13}+z^{12}+z^{11}+3z^{10}+5z^9+5z^8-2z^7-6z^6-z^5-z^4-3z^3-z^2-z-1)h(z^4)\\ +(-z^{13}-z^{12}+3z^{11}+3z^{10}-4z^7-4z^6+z^5+z^4+z^3+z^2)h(z^8)\\ +(-2z^{13}-2z^{12}+2z^{11}+2z^{10})h(z^{16})=0,\end{aligned}$$

which is a 2-Mahler equation satisfied by h.

6.5. **Carlitz zeta-function.** Let us briefly present an example in positive characteristic. Let $p \geq 0$ be a prime number, θ be a transcendental element over $\mathbb{F}_p$ and $\mathbf{K}$ denote the complement of the algebraic closure of $\mathbb{F}_p((\theta^{-1}))$ for the valuation associated to θ^{-1}. As mentioned in Section 2, the values at 1 of the Carlitz zeta-function, ζ_C is equal to $f(\theta)$ where $f(z) \in \mathbf{K}[[z]]$ satisfies

$$f(z^p) + (z^p - \theta)f(z) = -(z^p - \theta)\,.$$

Thus, $f(z)$ satisfies the following Mahler equation, of order 2:

$$(z^p - \theta)(z^{p^2} - \theta)f(z) - (z^p - \theta - 1)(z^{p^2} - \theta)f(z^p) - (z^p - \theta)f(z^{p^2}) = 0\,. \tag{34}$$

One easily checks that the Newton polygon associated to this equation has only one slope which is null. Thus $\nu_\Theta = \nu_P = \nu = \mu = 0$. The matrix M is then given by the constant term of the inverse of the matrix of this system, that is

$$M = \begin{pmatrix} 1+\theta^{-1} & -\theta^{-1} \\ 1 & 0 \end{pmatrix}$$

Thus, it acts as an isomorphism on $V_0 := \mathbf{K}^2$. Then, Algorithm 1 returns

$$\overline{P} = \begin{pmatrix} 1 & 1 \\ 1 & \theta \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}.$$

Algorithm 2 returns $H = \mathrm{I}_2$ and Algorithm 3 returns

$$e_C = \begin{pmatrix} 1 & 0 \\ 0 & e_\theta \end{pmatrix}$$

Thus, a basis of solutions of (34) is given by $f(z)$ and $g(z)e_\theta$ where where $f(z)$ is defined as above and $g(z)$ satisfies

$$(z^p - \theta)(z^{p^2} - \theta)g(z) - \theta(z^p - \theta - 1)(z^{p^2} - \theta)g(z^p) - \theta^2(z^p - \theta)g(z^{p^2}) = 0\,.$$

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